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Physios. — "*Bands in the secondary spectrum of hydrogen*". By
G. H. DIEKE. (Communicated by Prof. P. EHRENFEST).

(Communicated at the meeting of May 31, 1924).

Among all the molecular spectra the secondary or many-lined spectrum of hydrogen occupies a special place. Owing to the small moment of inertia of the hydrogen molecule the distance between the lines in one band must be much greater than in any other band spectrum and for the same reason every band can contain but few lines in this case. This is the reason why the aspect of the many-lined spectrum is totally different from an ordinary band spectrum. It is impossible to recognize the bands at a glance, as is possible in most molecular spectra, and between the series of lines in one band generally lie many lines of a different origin.

In 1912 G. S. FULCHER¹⁾ found certain groups of lines with a constant difference of wavelength, and he demonstrated that these lines are strongest with low voltage discharges. He further pointed out that according to the measurements of DUFOUR²⁾ they do not show the ZEEMAN effect. F. CROZE³⁾ arranged a number of lines without the ZEEMAN effect in series, in which the second differences of the frequencies were nearly constant. The FULCHER bands form part of his series. LENZ⁴⁾ explained this regularity on the principle of the quantum theory of band spectra. In this paper we shall try to arrange further lines in bands and explain the origin of these bands. The results however lead to an interpretation differing from the one given by LENZ.

The secondary spectrum of hydrogen has in recent times been measured by WATSON⁵⁾, PORLEZZA and NORZI⁶⁾, CROZE³⁾, and most

¹⁾ G. S. FULCHER Phys. ZS. 13 p. 1137, 1912.

Astrophys. J. 37 p. 60, 1913.

²⁾ DUFOUR Ann. Chim. et Phys. 9 p. 361, 1906.

Journ. de Phys. 8 p. 258, 1909.

³⁾ F. CROZE Ann. de phys. 1 p. 35, 1914.

⁴⁾ W. LENZ, Verh. D. Phys. Ges. 21. p. 632, 1919.

A. SOMMERFELD, Atombau u. Spektrallinien, 3. Aufl. p. 534.

⁵⁾ H. E. WATSON Proc. Roy. Soc. A 82 p. 189, 1909.

⁶⁾ C. PORLEZZA e G. NORZI Atti Accad. Lincei 20 (1) p. 819, 1911.

C. PORLEZZA; ibid. 20 (2) p. 176, 1911.

exhaustively by MERTON and BARRATT¹⁾. The two last-named scientists also investigated the change of intensity at different conditions of discharge. The regularities found by FULCHER can easily be extended with the table given by them. We shall first mention the results found, and then discuss the admissibility of the conclusions given here.

Three groups of bands could be found, one in the red, one in the green, and one in the blue part of the spectrum. The red group is the most complete, and will therefore be taken as an example. The structure of it can be seen from the figure in which the number m is plotted as ordinate against the frequency as abscissa. The branches A , B , C , etc. are zero branches, that is to say, they are due to the transition $m \rightarrow m$ of the rotational quantum-number m , and the lines of these branches satisfy the equation

$$r = A + Cm'^2 \quad (m' = 1, 5, 2, 5 \dots) \quad (1)$$

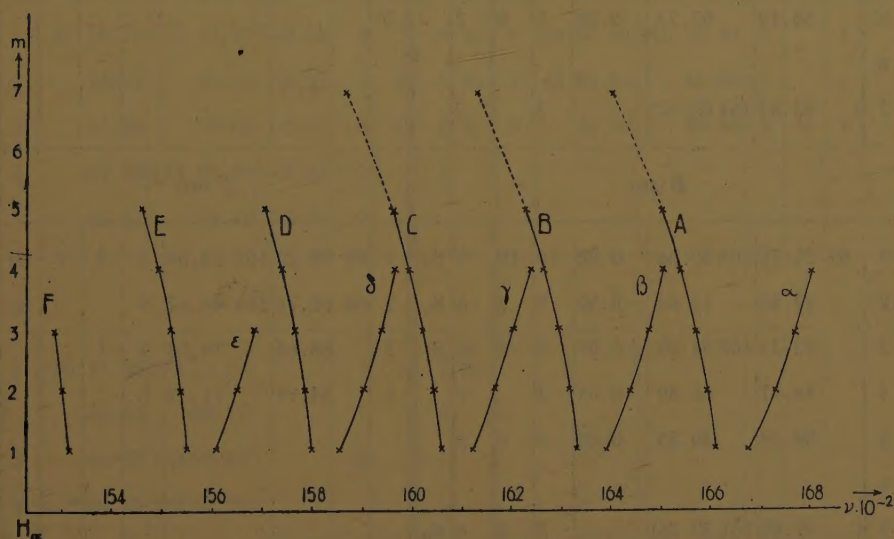


Fig. 1.

The branches marked with Greek characters may be taken for positive branches, and are therefore due to transitions $m + 1 \rightarrow m$. Whether a negative branch is present, could not be established with any certainty (see later on). In the fig. the six bands of the red group have been drawn. Every band, which in this case contains two branches, is characterized by the oscillation quantum numbers n_1 and n_2 of initial and final state of the molecule. In the group $n_1 - n_2$ is constant. The groups in green and blue have an analogous structure, but are less complete.

1) T. R. MERTON and S. BARRAT Phil. Trans. A 222 p. 369, 1922.

In the following tables I, II, III the wavelengths and frequencies of the lines are given according to the measurements of MERTON and BARRATT and also the intensity according to MERTON and BARRATT (M), WATSON (W) and PORIEZZA and NORZI (P).

TABLE I. Red group.

| m | λ Å | ν | $\Delta\nu$ | Intensity | | | F | m | λ Å | ν | Intensity | | | F |
|-------|-------------|------------|-------------|-----------|----|----|----------------|--------------|-------------|-----------|-----------|---|---|----------------|
| | | | | M | W | P | | | | | M | W | P | |
| A (m) | | | | | | | | α (m) | | | | | | |
| 1 | 60 18.29 | 166 11.43 | 0.00 | 10 | 9 | 7 | S ₃ | 1 | 59 94.05 | 166 78.59 | 6 | 4 | 4 | S ₄ |
| 2 | 23.74 | 165 96.41 | -0.01 | 7 | 4 | 4 | S ₂ | 2 | 75.43 | 167 30.58 | 10 | 8 | 8 | S ₇ |
| 3 | 31.80 | 74.14 | +0.11 | 10 | 5d | 10 | S ₁ | 3 | 59.70 | 74.72 | 8 | 3 | 3 | |
| 4 | 42.70 | 44.32 | -0.09 | 5 | | | | 4 | 47.27 | 168 09.80 | 6 | 3 | 2 | |
| 5 | 56.10 | 07.73 | 0.00 | 5 | 0 | 2 | | 5 | | | | | | |
| 6 | | | | | | | | | | | | | | |
| 7 (| 93.83 | 164 05.52) | | 2 | | 2 | | | | | | | | |
| B (m) | | | | | | | | β (m) | | | | | | |
| 1 | 61 21.76 | 163 30.66 | 0.00 | 10 | 10 | 9 | S ₃ | 1 | 60 98.23 | 163 93.66 | 6 | 3 | 2 | S ₄ |
| 2 | 27.40 | 15.64 | -0.50 | 9 | 6 | 6 | S ₂ | 2 | 60 80.78 | 164 40.52 | 9 | | | S ₇ |
| 3 | 35.34 | 162 94.53 | +0.05 | 8 | 6 | 10 | S ₁ | 3 | 66.64 | 79.03 | 4 | 1 | | |
| 4 | 46.17 | 65.80 | -0.04 | 3 | | 2 | | 4 | 54.74 | 11.43 | 0 | | | |
| 5 | 59.58 | 30.39 | 0.00 | 4 | 0 | 3 | | | | | | | | |
| 6 | | | | | | | | | | | | | | |
| 7 (| 97.05 | 161 32.25) | | 5 | 2 | 4 | S ₆ | | | | | | | |
| C (m) | | | | | | | | γ (m) | | | | | | |
| 1 | 62 24.81 | 160 60.31 | -0.02 | 9 | 10 | 9 | S ₃ | 1 | 62 01.15 | 161 21.59 | 5 | 2 | 3 | S ₄ |
| 2 | 30.23 | 46.34 | +0.01 | 7 | 2 | 3 | S ₂ | 2 | 61 82.98 | 68.96 | 8 | 4 | 6 | S ₇ |
| 3 | 38.39 | 25.35 | +0.02 | 8 | 6 | 7 | S ₁ | 3 | 67.74 | 162 08.93 | 3 | 0 | 1 | |
| 4 | 49.15 | 159 97.74 | -0.07 | 3 | 0 | 1 | | 4 | 55.61 | 40.85 | 5 | 0 | 3 | |
| 5 | 62.49 | 63.68 | +0.05 | 4 | | | | | | | | | | |
| 6 | | | | | | | | | | | | | | |
| 7 (| 99.42 | 158 70.09) | | 7 | 5 | 6 | S ₆ | | | | | | | |

TABLE I. (Continued).

| m | λ JÅ | ν | $\Delta \nu$ | Intensity | | | F | m | λ JÅ | ν | Intensity | | | F |
|--------------------------------------|--------------|------------|--------------|-----------|---|---|----------------|-------------------|--------------|-----------|-----------|---|---|----------------|
| | | | | M | W | D | | | | | M | W | D | |
| D (m) | | | | | | | | δ (m) | | | | | | |
| 1 | 63 27.07 | 158 00.73 | 0.00 | 8 | 8 | 7 | S ₃ | 1 | 63 03.46 | 158 59.93 | 3 | 0 | 1 | S ₄ |
| 2 | 32.46 | 157 87.28 | +0.06 | 5 | 1 | 3 | S ₂ | 2 | 62 85.37 | 159 05.58 | ? | 4 | 4 | S ₇ |
| 3 | 40.57 | 67.10 | +0.01 | 6 | 2 | 4 | S ₁ | 3 | 71.31 | 41.24 | 3 | 2 | 2 | |
| 4 | 51.28 | 40.51 | +0.04 | 1 | | 1 | | 4 | 60.31 | 69.69 | [—] | | | |
| 5 | 64.64 | 07.46 | —0.09 | 2 | 0 | 2 | | | | | | | | |
| E (m) | | | | | | | | ε (m) | | | | | | |
| 1 | 64 28.10 | 155 52.41 | +0.03 | 5 | 2 | 4 | S ₃ | 1 | 64 04.01 | 156 10.91 | 3 | | 1 | |
| 2 | 33.47 | 39.43 | —0.02 | 2 | 0 | 1 | S ₂ | 2 | 63 87.87 | 50.36 | 1 | | 1 | |
| 3 | 41.50 | 20.05 | —0.06 | 3 | 0 | 3 | S ₁ | 3 | 72.19 | 88.85 | 3 | 0 | 1 | |
| 4 | 52.13 | 154 94.49 | —0.08 | 0 | | 1 | | | | | | | | |
| 5 | 65.22 | 63.12 | +0.14 | 0 | | | | | | | | | | |
| F' (m) | | | | | | | | | | | | | | |
| 1 | 65 27.35 | 153 15.93 | | 4 | 1 | 2 | S ₃ | | | | | | | |
| 2 | 32.62 | 03.57 | | 1 | | 1 | | | | | | | | |
| 3 | 40.53 | 152 85.07 | | 2 | | 1 | | | | | | | | |
| 4 | (*)50.60 | 152 61.56) | | | | 1 | | | | | | | | |
| *) Wavelength according to PORLEZZA. | | | | | | | | | | | | | | |

*) Wavelength according to PORLEZZA.

The column $\Delta \nu$ gives the differences between the frequencies observed and the frequencies calculated according to the equation (2). In the last column is given FULCHER's notation for the lines found by this investigator.

Reliability of the data.

The red group is the strongest, and the observations referring to it are by far the completest. Therefore, the constants for it could be determined with a much greater accuracy than for the two other groups. For the same reason, the probability that the lines have

been wrongly arranged is smallest in the red group and generally smaller in the zero than in the positive branches. Only an experimental

TABLE II. Green group.

[illegible]

TABLE III. Blue group.

| m | $\lambda \text{ \AA}$ | ν | Intensity | | |
|---|-----------------------|------------|-----------|----|---|
| | | | M | W | P |
| A (m) | | | | | |
| ¹⁾ 1 | 44 90.45 | 222 63.24 | 5 | 5 | 4 |
| 2 | 93.67 | 47.29 | 2 | 1 | 2 |
| 3 | 98.52 | 23.31 | 2 | 4 | 5 |
| 4 | 45 04.90 | 221 91.84 | [—] | | |
| 5 | 12.81 | 52.96 | 1 | 0 | |
| B (m) | | | | | |
| 1 | 45 54.13 | 219 51.97 | 4 | 3 | 3 |
| 2 | 57.32 | 36.61 | 2 | 2d | 2 |
| 3 | 62.24 | 12.96 | 2 | 2 | 2 |
| 4 | | | | | |
| 5 | 76.54 | 218 44.45 | 0 | | |
| C (m) | | | | | |
| 1 | 46 17.49 | 216 50.73 | 3 | 3 | 4 |
| 2 | 20.72 | 35.61 | 1 | 0 | 1 |
| ¹⁾ 3 | 25.39 | 13.76 | (4) | 2 | 6 |
| 4 | | | | | |
| 5 | (40.47 | 215 43.50) | [—] | | |
| D (m) | | | | | |
| 1 | 46 80.43 | 213 59.62 | 2 | 1 | 0 |
| 2 | (83.79 | 44.26) | 5 | 4 | 7 |
| 3 | 88.43 | 23.15 | 0 | | 0 |
| E (m) | | | | | |
| 1 | 47 43.40 | 210 76.06 | 3 | 1 | 2 |
| 2 | | | | | |
| 3 | 51.59 | 39.75 | 1 | 0 | 1 |
| ¹⁾ These lines show the ZEEMAN effect. | | | | | |

investigation can give certainty on the correctness of the classification. Lines which seem to belong to these bands, but which are still doubtful, have been placed between brackets. A number of lines, which must very probably be attributed to these or analogous bands, is omitted for the reason of being too doubtful. Because of the great number of lines, the existence of relations between wave numbers alone is of little value.

FULCHER and LENZ regarded the lines $A(m)$, $B(m)$, $C(m)$ etc., m being a constant, as belonging to one band. But accepting this explanation, one cannot find a reason for the existence of several bands with small characteristic differences between them.

Therefore the lines, which LENZ and SOMMERFELD regarded as belonging to the same oscillation, but to different rotational quantum numbers, are interpreted in this paper as being due to the same rotational transitions but to different oscillation jumps.

GEHRKE and LAU¹⁾ also investigated the many-lined spectrum and confirmed FULCHER's results, viz. that the lines in question are much more strongly excited with low velocity cathode rays, than with rays of high velocity. They were the first to observe the blue group, but propounded another conception of the structure of the groups. They, as well as FULCHER, pointed out that the regularity in the arrangement of the lines is specially manifested in the wavelengths and they therefore called these bands $\Delta\lambda$ bands.

It seems to me that most of the regularities found by CROZE must be attributed to chance.

KRATZER²⁾ deduced certain relations between the constants of a band system. The fact that these relations are satisfied here may be considered as a proof that the lines are correctly arranged and interpreted.

The fact that the first lines of every band satisfy a quadratic equation, suggests that the electron impulse in the direction of the line connecting the nuclei is zero. If the influence of the rotation on the motion of the electrons is neglected, the rotational term may be written

$$F(m) = B(m - \varrho)^2 - Bu^2(m - \varrho)^4 + \dots$$

in which $B = \frac{h}{8\pi^2 J} - an$ (J moment of inertia), ϱ the component of the electron impulse vertically on the line connecting the nuclei,

¹⁾ E. GEHRKE and E. LAU Berl. Ber. 1922 p. 453; 1923 p. 242.

²⁾ A. KRATZER ZS. f. Phys. 3 p. 289, 1920, Ann. d. Phys. 67, p. 127, 1922.

and $u = \frac{h}{4\pi^2 J \nu_0}$ (ν_0 frequency of the nuclear oscillations at very small amplitudes); B is thus depending on the oscillation quantum number n . The small value of the moment of inertia J causes u to be larger here than in most other bandspectra, and therefore a deviation from DESLANDRE's law will take place already for rather small values of m . If in a certain electron- and oscillation transition the rotational quantum is unchanged, we get the zero branch

$$Q(m) = A + C m'^2 - D m'^4 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which

$$C = B_1 - B_2, \quad D = B_1 u_1^2 - B_2 u_2^2 \quad \text{and} \quad m' = m - \rho.$$

The observations indicate that m' has the values 1,5, 2,5 as is also the case with the infra-red bands of HCl and the ultra-violet cyanogen bands. As the lines belonging to the transitions $0,5 \rightarrow 0,5$ do not appear, it would be obvious to give ρ the value of $-\frac{1}{2}$. There are however other likely possibilities, so that in this respect nothing can be deduced as yet. The interpretation of the structure of the bands given here is not affected by it.

In table IV the values A and C are given, computed from equation (2). The column $\Delta\nu$ in the tables I and II indicates that the lines are in agreement with the equation within the limits of experimental errors. The agreement would be still greater, if for every band an individual value D had been taken. As D must be computed from differences of differences a reliable value cannot be calculated from one band. In the red and green group the average of the computed values of the separate bands is taken. Thus we found for the red group $D = 0,00201$ and for the green $D = 0,004$. The latter value however is of no importance as in the green group the constants in every case were computed out of the first four lines.

In the blue group the data are so few that here the simple equation (1) was used.

If the branches indicated by Greek characters belong to the transitions $1,5 \rightarrow 0,5$, $2,5 \rightarrow 1,5$ etc. the difference between the corresponding lines of the positive branch and the nullbranch must be equal to

$$B_2 - \beta_2 + 2(B_2 - 2\beta_2)m' - \beta_2(6m'^2 + 4m'^4), \quad . \quad . \quad . \quad (3)$$

in which $\beta = u^2 B$.

The values of B_2 computed from this expression are given in the third column of table IV. They must be taken as provisional values, however.

TABLE IV.

| Band | A | B_2 | $-C$ | ΔC |
|-------------|-------------|-------|--------|------------|
| Red group | | | | |
| A | 166 19.90 | 33.68 | 3.769 | 0.122 |
| B | 163 38.85 | 31.75 | 3.647 | 0.128 |
| C | 160 68.24 | 30.71 | 3.519 | 0.126 |
| D | 158 08.35 | 30.00 | 3.393 | 0.134 |
| E | 155 59.74 | | 3.259 | 0.144 |
| F | 153 22.94 | | 3.115 | |
| Green group | | | | |
| A | 188 63.56 | 33.41 | 5.313 | 0.187 |
| B | 184 56.87 | 32.28 | 5.126 | 0.147 |
| C | 180 65.21 | 31.15 | 4.979 | 0.167 |
| D | 176 87.26 | | 4.812 | |
| E | 173 27.03 | | | |
| Blue group | | | | |
| A | 222 72.22 | | 3.99 | 0.12 |
| B | 219 60.72 | | 3.87 | 0.09 |
| C | 216 59.27 | | 3.78 | 0.13 |
| D | 213 67.83 | | 3.65 | 0.02 |
| E | (210 84.23 | | 3.63) | |

The values B_2 and C depend according to KRATZER's theory on the oscillation quantum numbers

$$B_2 = B_2^0 - \alpha_2 n_2 \dots \dots \dots (4)$$

$$-C = B_2^0 - B_1^0 - (\alpha_2 - \alpha_1) n_2 + d \alpha_1 \dots \dots \dots (5)$$

Here α_1 and α_2 are numbers proportional with u , which contain the coefficients of the potential development of the forces between the nuclei, and B_1^0, B_2^0 the values of B_1, B_2 when there are no oscillations of the nuclei; $d = n_1 - n_2$ is constant in one group.

The values of B_2 are not known with sufficient accuracy to test

the linear decrease of B , with n , with any certainty. The decrease is, however, in the right direction and of the correct order of magnitude. On the contrary, column ΔC in table IV shows that the linear decrease with n , of $-C$ really takes place. The small rate of the differences, which seems to be real in the red group, may be due to the fact that the terms of higher order in (4) and (5) are neglected. We can attach most value to the numbers of the red group, because all the constants in the two other groups could not be determined with the same degree of accuracy.

Intensity of the lines.

The sharp decrease of intensity is to be expected on account of the small moment of inertia of hydrogen, and is to a less extent also observed in the band spectrum of helium¹⁾. The tables show that the different investigators give greatly divergent intensities, and therefore a wholly regular course cannot be expected.

But there is one peculiarity, in which all investigations agree and which ought to be mentioned. The lines with an even value of m in the Q -branches are weaker than might be expected from the intensities of the preceding and the following line. From this fact one could come to the conclusion that the lines 1, 3, 5... of the zero branch might be connected more closely and also the lines 2, 4, 6... Something of this kind would occur, if the odd lines belonged to the rotational transitions $m - \frac{1}{4} \rightarrow m - \frac{1}{4}$ and the even lines to the transitions $m + \frac{1}{4} \rightarrow m + \frac{1}{4}$. But we will not insist on this question, the experimental data being not yet sufficient to decide with any certainty in favour of one or the other assumption.

Changes by external influences.

According to the measurements of DUFOUR and CROZE, the band lines have no ZEEMAN effect. A few exceptions from this rule in the blue group will have their origin in the coincidence with other lines.

The first lines of each band in the red group are weakened by a condensed discharge and enhanced by lower pressure in the discharge bulb. The further lines show a more irregular behaviour with changes of the conditions of the discharge. The lines of the

¹⁾ W. E. CURTIS Proc. Roy. Soc. A 101 p. 38. 1922.

A 103 p. 315. 1923.

²⁾ The values $m \pm \frac{1}{4}$ are also found in the spectrum of helium. A. KRATZER Z.S. f. Phys. 16. p. 353, 1923.

green and blue group behave in the same manner, but the observations are less complete.

A much more regular progression of intensities than in a discharge bulb, where the conditions are too complicated, must be expected, if the lines are excited by temperature radiation in an electric furnace. With high temperatures, the higher lines in a band must be enhanced and more lines will probably appear. Also the intensity of bands with high oscillation quantum numbers must increase. It must be expected that these and analogous bands will be prominent in the spectrum if the temperature is not too high, and that the lines enhanced by a condensed discharge will be relatively weak. In this manner, it is possible perhaps to decide, whether the classification of the lines in this paper is correct and to find new bands with some certainty.

The zero lines of the bands in each group follow approximatively DESLANDRE's formula. We shall not insist now on a possible connection between the three groups. Nor the initial nor the final state of the bands can be the normal state of the molecule, and therefore a direct information on this state cannot be got from these bands. The bands originating in the normal state must, as may be seen from the resonance potential, lie in the extreme ultraviolet. Unfortunately the accuracy of the wave lengths measurements¹⁾ of the so called first spectrum of SCHUMANN is not sufficient to find such bands among the extraordinary great number of lines.

If the branches marked with Greek letters are interpreted as positive branches, as indicated above, the moment of inertia of the hydrogen molecule in the final state is of the order of magnitude $0,82 \cdot 10^{-40}$, whereas LENZ calculated $0,19 \cdot 10^{-40}$ from the FULCHER bands.

¹⁾ Th. LYMAN, *Astroph. J.* 23. p. 181, 1906.

Botany. — Communication on Miss A. BAKKER's „*Investigations regarding the existence of Separate Zones of Perception and Reaction in the Seedlings of Paniceae*”. By Prof. F. A. F. C. WENT.

(Communicated at the meeting of May 31, 1924).

Since ROTHERT performed, in 1896¹⁾, his well-known experiments on the phototropy of the seedlings of grasses, everyone is convinced that most grasses behave in this respect like *Avena*, that is to say, that in the coleoptile the sensitiveness to unilateral illumination is greatest at the apex, and gradually lessens from the apex downward, while the curvatures start at the tip and continue down to the base; meantime the top zones are gradually straightening again.

However, in seedlings of some *Paniceae* ROTHERT has discovered a different behaviour, more in accordance with the conception of CHARLES and FRANCIS DARWIN. It is well known that in *Paniceae* there is a distinct cotyledon with a hypocotyledonary axis. Now ROTHERT supposed that the cotyledon did not show any curvature and that, therefore, the bending power was localized only in the hypocotyledonary axis, while on the other hand he believed this axis to be insensitive to a unilateral light stimulus, so that in other words the sensitivity to unilateral illumination would be localized in the cotyledon. There would accordingly be a complete separation between the percipient zone: the cotyledon, and the reacting zone: the hypocotyledonary axis.

When Miss A. BAKKER was going to make experiments in my laboratory on the photogrowth reaction of seedlings of *Paniceae*, I deemed it expedient that she should first repeat the experiments of ROTHERT alluded to above. To our great surprise she did not succeed in confirming ROTHERT's results, but it appeared after all that the behaviour of seedlings of *Paniceae* does not in any way differ from those of *Avena*. I here wish to give provisionally a brief account of her experiments.

Miss BAKKER worked chiefly with seedlings of *Setaria viridis*; also with *Panicum sanguinale* and *Panicum miliaceum*. The seedlings

¹⁾ W. ROTHERT, COHN's Beiträge zur Biologie der Pflanzen VII, 1896.

were raised in zinc dishes at a temperature of 22° C. in the dark room of the laboratory greenhouse of the Utrecht Botanical Garden. After some four days they were fit for the experiment, having reached a length of 2—4 cm. They were much less evenly developed than seedlings of *Avena*.

The observations took place in the dark room of the Botanical Laboratory at about 18° C. Neither lightgas nor other laboratory gases are present in the air of this room. When seedlings of *Setaria viridis* are exposed for 10 seconds to unilateral light of about 100 M.C., all of them display positive curvatures (about 400 seedlings were examined). This curvature starts in the apex of the cotyledon and is distinctly to be observed after 55 minutes without any magnification; the maximal curvature of the cotyledon is distinguishable 10 minutes later; after 5 to 15 minutes the curvature has already passed on to the hypocotyledonary axis, while the cotyledon becomes straight again.

This is represented photographically in figs. 1 and 2, in which

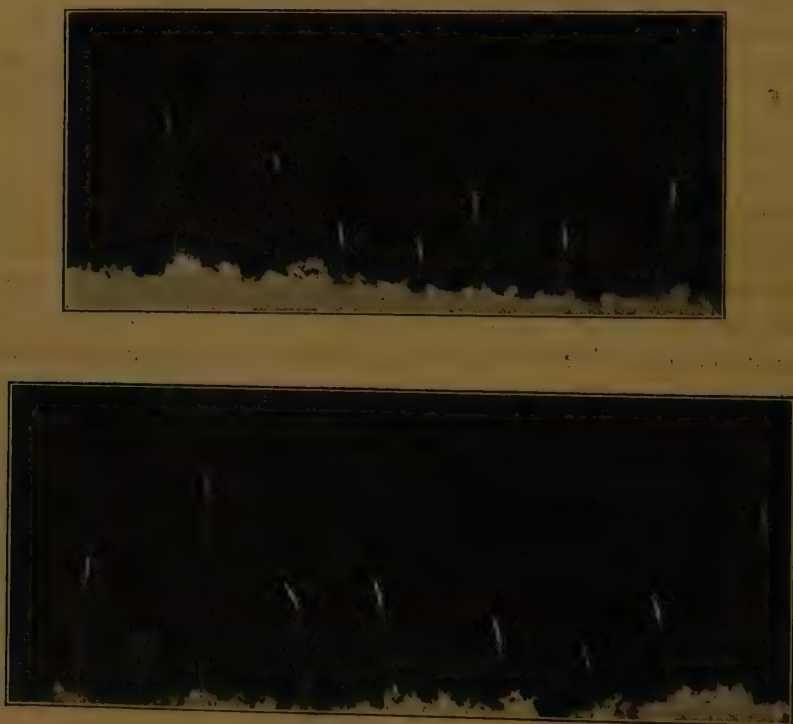


Fig. 1 and 2. Etiolated seedlings of *Setaria viridis* illuminated from the left for 10 seconds with 100 M.C., and photographed one hour later. The curvature is still partly visible in the cotyledon, partly also in the hypocotyledonary axis.

we see, in some plants, the curvature still located in the cotyledon, whereas in others a curvature is already observable in the hypocotyledonary axis. ROTHERT himself reports in the treatise mentioned on page 69, that there are solitary instances, in which in young seedlings the cotyledon may bend, but this fact did not strike him as being remarkable. For that matter the above goes to show that the curvature moves on very rapidly. The chances, therefore, are that the initial stages may be overlooked, unless the observer is constantly watching the process. Also in the case of *Panicum sanguinale* and *Panicum miliaceum* 100 % of the seedlings bent positively; but here the curvature of the cotyledon became noticeable only after two hours.

In order to ascertain whether the hypocotyledonary axis is sensitive to unilateral illumination the cotyledon was covered with a little cap of tinfoil, the lower rim of the cap overlapping the boundary between cotyledon and hypocotyledonary axis by 1—1½ mm.

These seedlings were illuminated unilaterally by a very strong lamp; it thereby appeared that after three hours no curvature was noticeable, but after 24 hours 45 of the 51 seedlings had bent positively (i.e. 80%), 5 of them displayed a negative curvature and one was erect.

Fig. 3 is a photo of such a seedling illuminated from the left.



Fig. 3. Etiolated seedling of *Setaria viridis*, the cotyledon being enclosed in a tinfoil cap, illuminated from the left by a strong electric lamp, and photographed after 24 hours.

A strong positive curvature of the hypocotyledon is to be seen.

FITTING has already observed something like this, as he reports that 17—25 % of his seedlings with caps bent positively¹⁾, but apparently he disregarded this phenomenon, thereby causing the erroneous opinion of ROTHERT to retain its influence in the scientific world, as becomes apparent from the literature.

Surely Miss BAKKER's experiment is of some interest in setting forth that also in the seedlings of Paniceae there is no question about separate percipient and reacting zones. This does away with one of the few remaining supports of the theoretical conceptions of PFEFFER and his followers, who will apply the principles of human

¹⁾ H. FITTING, Pringsh. Jahrb. f. wiss. Bot. XLV 1908, p. 110.

physiology to stimulation in plants. In my judgment they had rather follow the other way by carrying out a minute and unbiassed investigation of the much simpler stimulation in plants in order to utilize the observed facts to obtain a better insight into the stimulation phenomena in animals.

It is hardly necessary to say that the facts observed by Miss BAKKER agree with BLAAUW's conceptions regarding the origin of the phototropic reactions. True, also the facts generally recognized may be brought into accord with BLAAUW's idea, but then working-hypotheses are required that we can do without now.

Utrecht, May 1924.

From the Botanical Laboratory.

Colloid-chemistry. — "*Röntgen-spectrographical Researches on the Nature of the Swelling of Substances Giving a Fibre-diagram.*"

By Dr. J. R. KATZ. (Communicated by Prof. A. F. HOLLEMAN).

(Communicated at the meeting of March 29, 1924).

1. Purpose of the investigation.

The application of a new method of research has enabled colloid-chemists and biologists to answer the question which had been so often discussed by them, what is the nature of so called swelling or imbibition¹⁾ with far more certainty than had been the case so far. For the question whether the water is received between micells (small crystals or other corpuscles of a size ranging from one to a hundred $\mu\mu$) intermicellarly, or as a solid solution intermolecularly can be tested directly by the experiment by application of Röntgenspectrography, (in those swelling bodies that are crystalline). For if the swelling takes place intermicellarly, the crystal lattice will not change, whereas if the swelling is owing to the formation of a solid solution, the dimensions of the lattice must become larger.²⁾ It is known that by swelling is understood the reception of a liquid into a solid body, in which this seems to remain homogeneous, increases in size ("swells"), and diminishes in cohesion (stretches more, tears through a smaller force)³⁾. Almost all substances which are derived from animal or vegetable organisms, can swell. Particularly suitable for such Röntgenspectrographical researches seem to be fibrous substances, as according to known, often confirmed investigations by VON HÖHNEL, these bodies only swell as regards breadth and thickness (and often even strongly), whereas their length hardly varies at all. An increase of size of the lattice were, accordingly, only to be expected in those directions that are at right angles to the fibre axis. Röntgenspectrography has taught that fibres give a characteristic diagram consisting of line- or point-shaped spots, having the fibre axis and the direction normal to it as symmetry axes (so-called fibre-diagram or four-point diagram), and that the occurrence of such a diagram always means that the body is built up of many small

¹⁾ Cf. J. R. KATZ, *Gesetze der Quellung*, Kolloidchem. Beihefte 9, p. 1 (1916).

²⁾ Compare also L. VEGARD's researches on chabasite (*Ann. d. Phys.* **54**, p. 160—164), where the water *does* enter the lattice, but does not enlarge it.

³⁾ *Loc. cit.* p. 9—14.

elements, probably small crystals, all lying with one direction parallel to the fibre axis¹). POLANYI²) has shown that the spots in a fibre diagram lie on a set of hyperbolae, all having the same axes, and that the size of the lattice in the directions which are vertical to the fibre axis, expresses itself in the position of the interference spots, which *are situated on the equator* (the axis perpendicular to the fibre axis). It follows from this that the investigation in the case of fibrous substances is simplified to the measurement of the distance between symmetrical interference spots on the equator, in moist and in dry condition. If the water enters the lattice, the distance between each pair of these lines will become smaller³). If no change of these distances takes place, the imbibition of liquid, which causes the "swelling", must take place intermicellarly; this latter conclusion is, however, imperative only when the substance that gives the Röntgenspectrum, is also that which swells; whereas a positive result were in itself conclusive that at least part of the swelling takes place intermolecularly. But it is probable just in the fibrous substances — as I will set forth later in another paper — that the substance which swells is the same as that which gives the Röntgenogram.

Later on I shall describe investigations on swelling bodies with another structure than fibre structure.

2. *The practical execution of the experiments.*

As source of light served a SIEGBAHN-HADDING Röntgentube⁴) with copper anticathode with three windows, which were closed with thin aluminium foil. It was fed by an intermittent continuous current of 50—60 kilovolts tension, which was obtained by means of a high-tension transformer with rotating rectifier. The strength of the current in the tube was 15—20 milliamperes. The exposures lasted from eight to fifteen hours, depending on the thickness of the fibre bundles. That such long exposures were required was owing to this that the aperture of the diaphragm and the thickness of the fibre bundles were chosen small; in consequence of this thin interference lines were obtained, which could be measured very sharply.

¹) HERZOG and JANCKE, Ber. d. deutsch. chem. Gesellsch. 53, p. 2162 (1920); HERZOG, JANCKE, POLANYI, Zeitschr. f. Physik, 3, p. 343 (1920).

²) Naturwissenschaften 9, p. 339, gives a summary of his views.

³) Unless it *does* enter the lattice, but does not enlarge it, as L. VEGARD has found in the zeolite chabasite, but then the "swelling", the increase in size of the body, is not explained.

⁴) Described by A. HADDING, Zeitschr. f. Physik, 3, p. 369 (1920).

The Röntgenlight transmitted through the aluminium windows and the paper layers was practically monochromatic. As appeared from photos with a rock-salt crystal, it consisted for by far the greater part of K_{α} -radiation of copper, K_{β} being present by the side of it in a much smaller quantity. Besides the characteristic radiation there was some "white" Röntgenlight. But on the diagrams obtained with comparatively still short exposure only the lines of the K_{α} -radiation are often visible; its wave-length is 1.54 \AA.U. as is known.

As the measurement of the distance between fibre bundle and plate is easily attended with relatively great errors, the moist and the dry substance were photographed *successively in the same camera*. The drying of the moist substances took place in the camera itself, so that the fibre bundles need not be touched between the two photographs.

The camera was constructed of brass plate covered with lead. The diaphragm consisted of a cylinder 60 mm. long with a round bore of 1.5 mm. diameter (or 40 mm. long and of a diameter of 1.0 mm.). The back side had a wider bore, so that the back rim of the opening could not be struck by the beam of the rays. The front side of the diaphragm was shut off with black paper, which was covered with a layer of sharply dried collodion in order to render the closure impenetrable to watervapour, or closed with a thin aluminium leaf (fixed fast with picein).

The back side of the diaphragm, a larger round plate of nickelled copper, had a very accurately centred slit; a fibre bundle could be centrically glued in this slit (with wax). It is important that the bundle passes exactly through the centre of the bore of the diaphragm.

The plate was at right angles to the bore of the diaphragm. In general Agfa-Röntgen plates were used — size $9 \times 9 \text{ cm.}$ They were placed in the camera wrapped in black paper. A lid of black lacquered brass plate with metal springs held the plate accurately in its place. It was pressed on all four sides against a rim of brass plate, covered with a thin felt strip. Though felt is a swellable body, like the paper in which the plate was wrapped, it was practically at the same distance from the fibre bundle with moist and with dry substances; the difference could be neglected. The wrapping up in paper protected the layer of gelatine of the plate against a too great absorption of water; if this however, did take place — which was sometimes the case in too protracted experiments — it seemed to me that mostly a rather inconvenient dimming of the moist plate ensued. In experiments of ordinary duration the paper

was still a sufficient protection. Rodinal 1:8 was used for the development.

In all experiments a screen was used of a leaden plate 2 mm. thick, which surrounded the camera, and transmitted only the diaphragm. This prevented the appearance of disturbing shades and cloudiness in consequence of false light. The metal lid behind the plate had (at the place where the central spot appears) a round aperture of 15 mm. diameter; it was glued to with black paper, and covered with several layers of collodion, which were sharply dried.

The course of an experiment was as follows: First a swollen substance (obtained by one or two days' swelling in a suitable liquid, here water) was stretched out to a bundle of fibres directed exactly parallel, on the back side of the diaphragm in the centred slit. The camera contained a large quantity of very wet filter paper (which was kept in immediate contact with the fibre bundle, so that this remained moist also through capillary action). The plate with paper envelope had been in the camera already some hours previously, so that it could not have a drying influence any longer. The lid was closed hermetically by means of a leucoplast band. Then the plate was exposed for from eight to fifteen hours. Great importance was attached to this that the photographing of the moist substances went on uninterruptedly, and was completed the very same day. There might else be reason to fear that on account of the so-called VON SCHROEDER-effect¹⁾ the degree of imbibition of the examined substance might become considerably smaller during the experiment, so that it would no longer be in agreement with that of saturated substance. During the photographing *particular attention had to be paid to the centring* (light spot on a light screen behind the camera as intensive as possible); else a very slight efficiency of the illumination may be counted on (focal spot 1—1½ mm., diaphragm 60 mm. long and 1½ mm. diameter).

As soon as this first photograph was ready, the wet filter-paper was removed from the camera, and this was dried by a short exposure to the air. Then some glass dishes with phosphorous pentoxide were placed in the camera, a plate with an envelope of black paper of the same thickness was inserted, and — after closure with the lid with the leucoplast band — the whole was placed in a thermostat of 40° C. At this temperature the drying goes much quicker than

¹⁾ VON SCHROEDER, Zeitschr. f. physikal. Chemie, **45**, p. 109 (1903); H. FREUNDLICH, Kapillarchemie, 2nd ed., p. 925.

at room-temperature. The next day the phosphorus pentoxide was renewed, the camera was again brought at 40° C.; the next day it was ready for the photographing. It was important that first the moist, and then the dry photo was taken. For it is preferable to examine as high a degree of imbibition as possible; and a swelling substance, which has once been sharply dried, can mostly no more imbibe as much water as before.

I have endeavoured to take the two photographs with equally strong exposure. Unfortunately it was not always possible to obtain negatives which were exactly equally dark. It was, indeed, approximately possible; and this was sufficient.

In measuring the negatives the middle of the interference lines was determined, and then the distance was measured of this middle from each pair of equatorial bands lying symmetrically. In the following tables these distances are given in mm. Each value recorded there is the mean of four separate measurements. The values given are accurate down to a few tenths of millimeters. The measuring took place with a transparent glass measuring rod divided into mm. on an illuminated piece of milk glass as background ¹⁾.

It was of little importance for my purpose to measure the distance between the fibre bundle and the photographic plate accurately. The chief point was that it was exactly the same with moist and with dry substances. Nevertheless I have every time determined the distance, but only with an accuracy of $\pm 0,7$ mm. The measurement took place with a thin metalwire. The length of the piece of wire was determined, which reached from the front end of the diaphragm to the plate, resp. to the back end of the diaphragm; the difference between these two values was the distance between plate and fibre bundle. If a more accurate determination of this distance should be required, a substance with accurately known lattice-constants might be placed in the camera, and the distance might be calculated from the situation of the interference bands. The distance used generally lay between 45 and 60 mm. With the cameras described here (fastening of the fibre bundle on the backside of the diaphragm) it is technically an easy matter to obtain good sciagrams of fibrous substances at a HADDING tube; the employment of this Röntgen tube

¹⁾ In a later resumé of the facts now to be published in separate papers I hope to be able to publish also measurements of these negatives with a registering photometer; when the separate silver grains do not give too much trouble, a greater accuracy will then be possible in the measurement of the distances between symmetric inference bands.

is, however, attended with the known difficulties of the high vacuum technique.

3. *The swelling of cellulose and hydrate-celluloses in water.*

In the following tables — in which the distances between the equatorial bands lying symmetrically is recorded in mm. — the intensest interference band is printed in bold faced type; this could generally be measured most accurately, so that these holdtype values allow of the most trustworthy comparison. It corresponds with the plane (020) of the cellulose¹⁾. All the water-percentages are given in ordinary parts of water per 100 ordinary parts of dry substance. In this first paper I have confined myself to making estimations about the value of the water percentages; I have done so by determining how much water vapour the substances had imbibed over 5 % sulphuric acid. Control by comparison of the cross-sections of the fibre moist and dry follows later.

a. Ramie fibres, cotonized, not mercerized; normal cellulose diagram ²⁾).

| Moist | Dry |
|-------------|-------------|
| 30.1 | 30.3 |
| 40.5 | 40.6 |
| 45.5 | 45.3 |

The moist substance contained $\pm 14\%$ water. As according to VON HÖHNEL the swelling only takes place normal to the fibre axis, the substance in moist condition ought to have shown interference bands lying 7 % less far apart with intermolecular imbibition, than in dry condition. This appears not to be the case; no difference can be detected.

b. Ramie fibres, feebly mercerized (one minute under tension in 17 % NaOH); the substance presents the typical diagram of the

¹⁾ According to POLANYI's formula (cf. *Naturwissenschaften*, 9, p. 288), the cellulose would consist of rhombic crystals; the longest axis (long 10,2 Å.U.) then lies parallel to the fibre axis, the two other axes measuring 7,9 and 8,45 Å.U. Then the cellulose would have no high molecular weight, in contradiction to all that has been assumed up to now.

²⁾ Ramie is the cellulose fibre most suitable for such experiments, because these fibres do not show a spiral shaped ordering of the crystallites as so many others, but a parallel one.

mercerized cellulose, the line of (200) is equally intense as of the plane (020), whereas with the unchanged cellulose the former is many times weaker than the latter¹⁾.

| Moist | Dry |
|-------|------|
| 26.6 | 26.9 |
| 36.6 | 35.9 |
| 40.9 | 41.0 |

Nor are there any indications here that the water enters the lattice. The water-content of the moist substance was $\pm 18\%$. The change of the lattice should in that case have amounted to 9% .

c. *Manilla hemp*, light yellow, almost white. Normal cellulose diagram. This substance was examined, because according to WIESNER, it imbibes exceedingly much water; he states up to 50% , which, however, was not confirmed in the samples examined by me (nor is it, indeed, probable, in view of its use as ship's ropes).

| Moist | Dry | lines |
|-------|------|----------------|
| 36.2 | 35.6 | 60.5 ± 0.8 |
| 51.4 | 51.4 | |

The water content was $\pm 18\%$. Yet no change of the equatorial interference bands could be shown, though a change of 9% might be expected in the case of intermolecular imbibition.

d. *Artificial silk from viscose*. (Vistra artificial silk of the Köln-Rottweil A. G.). Like some other artificial silks of the German industry this kind of artificial silk presents a four-point diagram, former brands showing DEBYE-SCHERRER circles. This means that the industry has succeeded in orientating the crystallites parallel to a great extent, just as is the case in the natural fibres. The substance consists of so-called *hydrate cellulose*, and presents the characteristic diagram of cellulose mercerized with sodium hydroxide: (200) just as strong as (020).

¹⁾ The second of the interferences recorded could not be measured with certainty on account of false light on the plate.

The water content of the moist substance was $\pm 23\%$. There is an indication of a change in the lattice (enlargement on swelling)

| Moist | Dry | Distance |
|-------|------|----------------|
| — | 2.34 | 53.7 ± 0.8 |
| 38.2 | 38.7 | |
| 42.4 | 42.6 | |

of 1 to 2%, but it could not be proved with *certainty*. In each of the direction normal to the axis the increase of the dimensions on intermolecular swelling, ought to have reached at least $11\frac{1}{2}\%$.

e. Artificial silk from copper oxide ammoniac, 120 deniers (ADLER artificial silk of the Firm J. P. BEMBERG A.G.) again exhibits a beautiful four-point diagram. Here too the crystallites are directed parallel in a high degree, so that a structure is reached which comes very near to the structure of the natural fibre substances. The substance consists of *hydrate cellulose* and presents the same diagram as cellulose mercerized in sodium hydroxide, probably in consequence of the treatment with copper oxide ammoniac¹⁾. These negatives are reproduced in fig. 1 and 2 in exactly the original size; unfortunately enough such Röntgen negatives lose much in the reproduction.

| Moist | Dry (3 negatives) | | | Distance |
|-------|-------------------|------|------|----------------|
| 22.9 | 23.2 | 22.9 | 23.4 | 53.0 ± 0.8 |
| 39.6 | 39.4 | 39.5 | 39.6 | |
| 44.0 | 43.8 | 44.0 | 44.0 | |

A change of the lattice cannot be observed. The water content of the swollen substance was $\pm 23\%$. The moist substance has a broad amorphous ring as liquids and amorphous substances show it; the diameter of this ring amounted to 57.1 mm. This ring was not found in the dry substance. Its intensity seemed greater to me in the direction of the fibre axis than normal to it²⁾.

f. Ramie fibres, mercerized by heating in saturated zinc-chloride

¹⁾ Ramie swelled in copper oxide ammoniac and then washed shows an analogous diagram.

²⁾ I am engaged in the study of these and analogous rings, and hope soon to be able to make further communications on this subject.

ROENTGENSPECTRA OF HYDRATE CELLULOSE.

(Adler artificial silk).

moist.

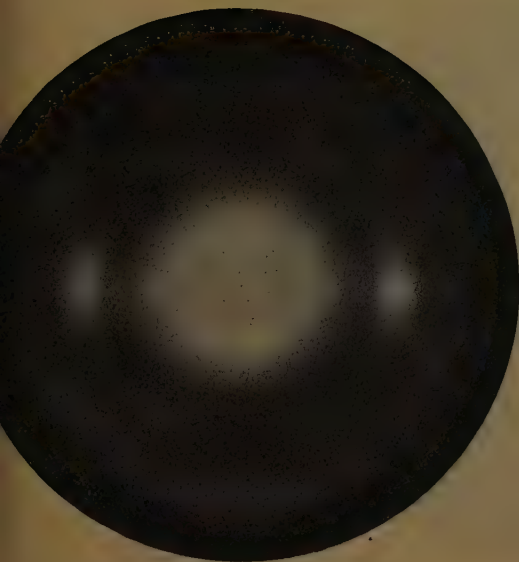


Fig. 1.

dry.

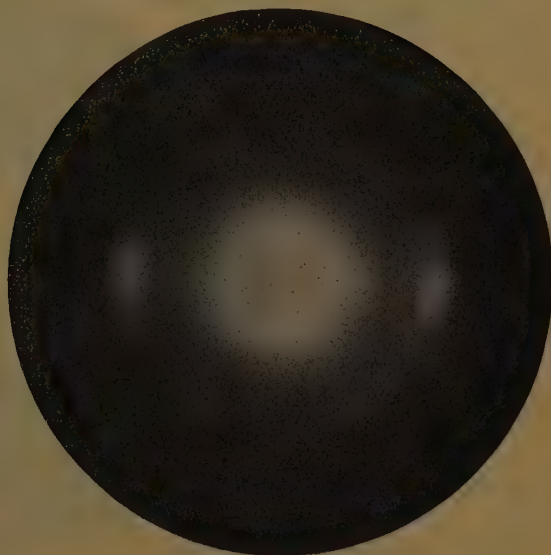


Fig. 2.

ROENTGENSPECTRA OF SILK FIBROIN.

moist.

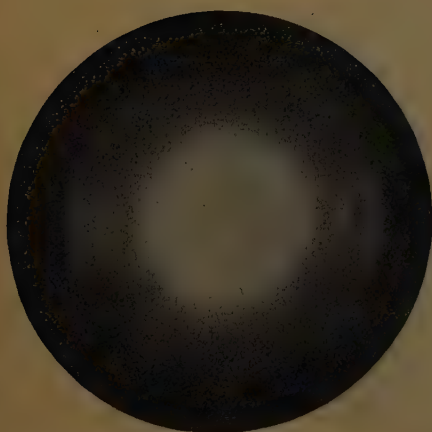


Fig. 3.

dry.

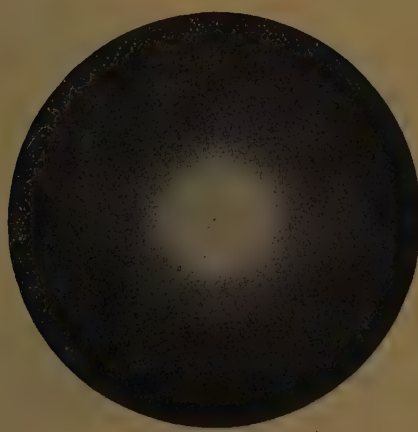


Fig. 4.

ROENTGENSPECTRA OF RAMIEFIBRES.

Without previous treatment.

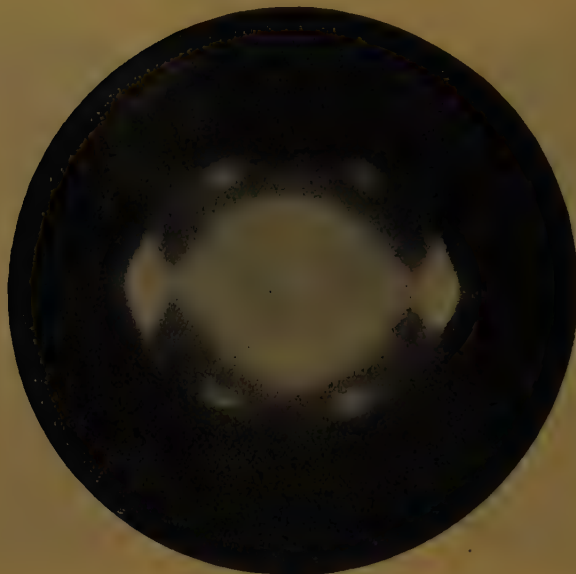


Fig. 5.

Washed out after treatment with 27 % NaOH.

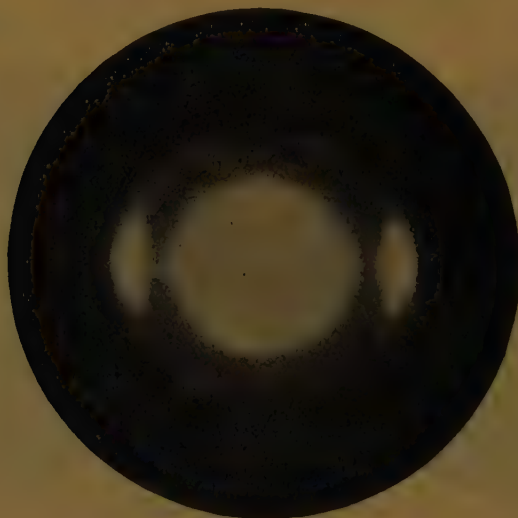


Fig. 6.

All this reproduced at the natural size.

solution (for 20 minutes at 100° C., then for 10 minutes at boiling-point temperature in zinc-chloride solution saturated at room temperature; a sample that had been boiled 10 min. longer, had quite gone into solution). The substance consists of *hydrate cellulose*; it presents, however, the normal cellulose diagram.

| Moist | Dry | Dry unchanged ramie | Distance |
|-------|------|---------------------|------------|
| | | | 47.5 ± 0.8 |
| 39.8 | 40.1 | 40.0 | |

The water content of the swollen substance was $\pm 18\%$. Yet no change of the lattice can be observed.

Summarizing it may be said that a difference in the dimensions of the Röntgenograms of swollen and dry celluloses cannot be shown with certainty, though when the swelling is intermolecular, displacements of the interference bands ought to have appeared exceeding the possible error of the experiments many times. If any water enlarges the lattice, this is only a small part of the total quantity imbibed; the changes in the lattice observed in one case do not lie, however, sufficiently far outside the margin of errors to be considered as convincing. If, therefore, the substance yielding the spectrum, is the principal imbibitive substance in the fibres, the *imbibition in all celluloses and hydrate celluloses examined is intermicellar*.

That in cellulose the imbibition is intermicellar is also made probable by the remarkable change of the diagram on prolonged swelling in strong (27%) sodium hydroxide. Fig. 3 is a diagram of unchanged air-dry ramie, fig. 4 of the same substance after the sodium hydroxide had been washed out (after it had acted on the ramie for 24 hours). The fibre diagram has almost become a DEBYE-SCHERRER diagram: the crystallites have lost their parallel arrangement; exactly what was to be expected in the case of very strong imbibition, when the swelling takes place micellarly, and the micells are identical with the crystallites. For why should they continue to be directed parallel, when through the swelling they recede more and more from one another?

The investigation is being continued with strongly mercerized celluloses, in which yet a small quantity of water perhaps enters the lattice.

It is noteworthy that dry and moist gave equally sharp diagrams.

4. The swelling of silk fibroins in water.

The diagram of silk fibroin has been extensively investigated by R. BRILL¹⁾. It has been found that the diagrams of the real and the wild silks present only insignificant differences, so that the crystalline substance in fibroin is probably identical in all these substances. It has, however, not yet been explained to what the small differences are owing²⁾.

a. *Stripped silk* of *Bombex mori* (*genuine silk*); a light brown preparation obtained by boiling out natural silk with a five-percentage solution of Marseilles-soap reacting purely neutrally. These negatives have been reproduced on the plate in fig. 5 and 6 in exactly the original size. The fibre diagram presents a first beginning of transition to a DEBYE-SCHERRER diagram.

| Moist | Dry | Distance |
|-------|------|---------------------------|
| 31.2 | 31.2 | (probably) 52.1 ± 1.0 |
| 39.1 | 39.1 | |

The water percentage of the moist substance was $\pm 15\%$. Hence the change of the equatorial interference bands had to be $7\frac{1}{2}\%$, because also silk does not swell in the direction of the fibre, but only normal to it. A change of the lines can, however, not be observed.

b. *Silkworm*; the substance is obtained by pressing out the contents of the silk glands of the silk worm in diluted acetic acid. This thread probably contains, therefore, besides crystalline fibroin, also amorphous sericin (silk gelatin), so that here certainly not all the swelling substance contributes to the formation of the spectrum. Transparent thread of 0.5 mm. diameter.

| Moist | Dry | Distance |
|-------|------|----------------|
| 42.1 | 41.9 | 61.0 ± 0.8 |
| 46.9 | 47.0 | |

Here the degree of imbibition of the swollen substance can be determined much better than in other cases, as the homogeneous,

¹⁾ R. BRILL, Diss. Berlin, 1923; Ann. d. Chemie **434**, p. 204 (1923).

²⁾ Oral communication by M. POLANYI.

fairly thick thread can be taken out of the water, dried, and then weighed (without water which is bound capillary, remaining hanging between the fibres as with fibrous substances). This degree of imbibition was 23 %. Yet no variation of the lattice can be detected.

c. *Fibroin from E. MERCK*; consists of shiny wound threads of a pure white. These were unrolled before the photo was taken, and photographed as a bundle of fibres directed parallel. The diagram presents two equatorial bands, much less sharp than with the two preceding fibroins.

Besides the moist substance shows a broad ring of a diameter of 53.5 mm., which is not found in the dry substance. This fibroin shows a much greater imbibitive power than any other fibroin preparation known to me.

| Moist | Dry | Distance |
|-------|------|----------------|
| 41.0 | 40.9 | 53.0 \pm 0.8 |

The moist substance contained $\pm 27\%$ water. It would, therefore, have to show a change of the equatorial bands of $13\frac{1}{2}\%$; there is, however, no change.

The diagrams for moist and dry were equally sharp in all these three cases.

Summarizing we may say that in the silk fibroins no change in the lattice of the crystalline substance is found when the substance swells in water. When, therefore, the substance which gives the spectrum, is the chief swelling substance (here already less certain; for which we refer to the sericin in the silkworm!) *the swelling of the silk fibroins must be intermicellar*, because otherwise the increase of the dimensions cannot be accounted for.

5. Does the lattice change in the direction of the fibres?

As VON HÖHNEL has shown and numerous investigators have confirmed that the length of the fibres does not perceptibly change (at most $\frac{1}{10}\%$), it is very improbable, indeed, that in the direction of the fibre axis any change in the lattice takes place, when this is absent in the directions normal to the fibre axis. This could only take place with organized bodies of complicated structure, in which the body is deformed through tensions in the longitudinal direction.

Hence we have no reason to believe that such complications are found in the substances studied in this investigation: fibres of cellu-

lose and of fibroin. Whether the lattice varies in the longitudinal direction might be most clearly decided in doubtful cases by oblique exposures¹⁾ (so that bundle of fibres and plate form angles with the horizontal beam of Röntgen rays). I have not considered the matter of sufficient importance to take such oblique photos, moist and dry.

For enough can already be derived from the vertical distance of the interferences of higher order in diagrams in which these are clear. If this vertical direction does not change (vertical = at right angles to the equatorial line), it may be concluded from this that the lattice does not change in the direction of the fibre axis either²⁾. The following values were found for the vertical distance of the central interferences on the hyperbolae of the first order (righthand interference band and lefthand interference band) in the above mentioned negatives:

| | Adler-artificial silk | | Ramie-fibres | | | Silkworm | |
|----------|-----------------------|-------|--------------|-------|---|----------|-------|
| | Dry | Moist | Dry | Moist | Moist, washed out after swelling in boiling ZnCl_2 | Dry | Moist |
| <i>r</i> | 35.0 | 35.0 | 30.0 | 30.5 | 30.0 | 29.5 | 29.9 |
| <i>l</i> | 35.0 | 35.0 | 30.5 | 30.0 | 30.0 | 29.7 | 29.8 |

Accordingly our Röntgen negatives show that the crystal lattice did not undergo any change in the direction of the fibres either (as was, indeed, to be expected).

CONCLUSION.

If the Röntgen spectrum rests on the repetition of the period of identity in the lattice of a microcrystalline substance, and when this substance is the chief substance that swells, — and this is probable, — the imbibition must be intermicellar in celluloses, hydrate celluloses, and silk fibroins, because else the "swelling" cannot be explained.

This result does not justify us in concluding that the imbibition would *always* be intermicellar. In the description of the experiments

¹⁾ See M. POLANYI and K. WEISSENBERG, *Zeitschr. f. Physik*, **9**, p. 123—130 (1922).

²⁾ M. POLANYI, *Faserstruktur im Roentgenlichte*, *Naturwissenschaften* **9**, p. 340 (1920).

with DEBYE-SCHERRER diagrams, which I carried out in conjunction with Dr. H. MARK, we shall meet with a swelling substance in which the period of identity of the lattice increases about in the same degree as the visible dimensions. It is, therefore, necessary to examine separately in every case of what nature the imbibition is.

When in the beginning of 1923 I had conceived the plan to study the problem of imbibition also Röntgenographically, I asked for and obtained hospitality in the Kaiser Wilhelm Institut für Physikalische Chemie und Elektrochemie and für Faserstoffchemie at Dahlem-Berlin for carrying out the first series of researches. This investigation on fibre-diagrams was performed there in the summer of 1923. When working out the fundamental idea, the experiences on the Röntgenspectroscopy of colloids and fibrous substances obtained in these laboratories, were readily put at my disposal. With great pleasure I express my indebtedness to Prof. F. HABER and Prof. R. O. HERZOG for the hospitality granted me, and also to Prof. H. FREUNDLICH for the unflagging warm interest shown by him in these researches. Dr. J. BÖHM and Dr. H. MARK have given me valuable assistance in the execution of these investigations.

Colloid chemistry. — "Röntgenspectrographic Researches on the Nature of Swelling in some Substances which Give a DEBYE-SCHERRER Diagram". By Dr. J. R. KATZ and H. MARK. (Communicated by Prof. A. F. HOLLEMAN).

(Communicated at the meeting of March 29, 1924).

1. Purpose of the investigation.

In an earlier paper one of us¹⁾ has described experiments on swelling substances yielding a fibre diagram. It then appeared that in these substances (cellulose, hydrate celluloses, and fibroin) no change of the crystal lattice takes place when they swell in water. To continue this research we have now in collaboration examined a number of other substances in the hope of finding one in which the crystal lattice does change. Before describing these experiments we will shortly consider what cases are *conceivable* when a strange substance penetrates into the lattice of a crystal, the former substance preserving its crystalline structure:

a. the elementary body receives a definite number of molecules of the penetrating substance, so that *for every degree of imbibition the elementary body possesses a definite stoichiometrical proportion of the two kinds of molecules*, together forming the lattice (hydrate formation or mixed crystal formation). In this case these entering new molecules take up a *fixed* place in the lattice. If the elementary body at first contained n molecules of the substance A , it now contains $nA + mB$, when m molecules of the substance B have penetrated into each elementary body. If the penetration is owing to the formation of a compound or of a hydrate in stoichiometrical proportions, m is generally a whole number (or a simple fraction?). If, however, a solid solution is formed, m need not necessarily be a whole number, as experiment has taught. This result was surprising. Also when the elementary body contains only a whole and small number of atoms, the volume of the elementary body appears to depend on the concentration according to a continuous function²⁾, a result that cannot very well be explained as yet³⁾. Possibly the

¹⁾ J. R. KATZ, Röntgen-spectrographical Researches on the Nature of the Swelling of Substances Giving a Fibre-diagram. These Proc., p. 505.

²⁾ Compare the researches of L. VEGARD and F. KIRCHNER.

³⁾ Compare WALTHER GERLACH, *Materie, Elektrizität, Energie*, TH. STEINKOPF, 1923, p. 163.

value found for the elementary body is a statistic middle value. Though great darkness still prevails about such variations of the crystal lattice through a penetrating substance, thus much may at least be said that when the density of the crystal has not been changed, the elementary body must have become larger. This will be expressed in the diffraction image in this way that the interferences *according to their position* are now represented by a quadratic form, in which other larger fundamental periods, possibly also other angles, appear. It is also possible that *the intensities* with which corresponding planes reflect, are different in the enlarged and in the not enlarged lattice; for in general the structure factor of a plane (hkl) will be changed by the appearance of new diffraction centres (atoms). This latter influence of imbibition can very well escape observation, as these variations of intensity need not be great.

It is, however, also possible that the density of a swollen substance d' and that of the dry substance are different. If the second substance is received into the lattice according to stoichiometrical proportions, if the molecules of the second substance get a fixed place in the crystal lattice (so that this is not simply enlarged by the second substance, which spreads irregularly through the elementary body or round the molecule), it can be ascertained that the volume of the elementary body has remained of exactly the same dimensions, when the ratio $\frac{d}{d'}$ is equal to $\frac{nA}{nA + mB}$. Here too it is to be expected that the intensities will in general become different. However, these variations too may likewise escape observation.

b. It is, however, also possible that the water of imbibition, when it penetrates the crystal lattice, is not received according to stoichiometrical proportions. In this direction points the perfect continuity of the change of all the properties as function of the degree of imbibition. Against this pleads, however, the fact that this is also the case in the experiments of VEGARD and of KIRCHNER on mixed crystals (isomeric mixtures). But on reception of liquids into swelling substances it is much more probable that in the case of an intermolecular imbibition the liquid molecules, without losing their mobility and *without occupying a definite position in the*

¹) When a real chemical substitution takes place (e. g. methylation of ureum), analogous views have already been published by P. GROTH (Elemente der physikalischen und chemischen Krystallographie, 1921, p. 278); they have led to the conception of "topical parameters" (compare e. g. H. MARK and K. MEISEL, Z. f. physik. Chemie 1924).

lattice, press apart the components of the elementary body, or else the separate molecules, so that the distance between them becomes the greater the more liquid has penetrated. The examination of the Röntgenspectrogram may then possibly give us an insight into how this takes place (whether e.g. it happens when an "adsorption" round each separate molecule¹⁾ or round definite groups of it takes place, and in what way this "adsorption" is related to the polyvalencies of the atoms of the lattice; or whether the old supposition is correct that there exist still larger units in a crystal than molecules; in the latter case the micells would be smaller than the smallest crystals). In this case we shall have to expect that the size of the lattice will be proportional about to the third power root of the volume. But it then will be necessary to show that really the dimensions of the lattice too are a continuous function of the degree of imbibition.

It appears from these considerations: from a change in size of the elementary body it may be concluded that the imbibition liquid enters the lattice of the swollen substance; in what way it is received into the lattice will have to be shown by further investigations²⁾. Inversely it cannot be said that when the volume of the elementary body has remained constant, the imbibition liquid has certainly not entered the crystal lattice³⁾.

2. *Description of the experiments.*

We have examined three substances which give diagrams that are equally sharp and comparable in all respects in swollen and in dry condition: inuline and chitine when swelling in water, and a soap, sodium palmitate, when swelling in trichlorethylene. Of the two substances swelling in water the isotherm had been measured before⁴⁾; the watervapour tension depends in both cases on the degree of imbibition according to a continuous curve. The isotherm has the S-form typical for bodies swelling in water. Hence it is certain that we have not to do with single hydrate formation, as

¹⁾ The term of "adsorption" is, strictly speaking, inappropriate here, as it only applies to polymolecular surfaces, not to binding to a single molecule.

²⁾ Compare also W. M. KEEHAN's investigations of palladium hydrogen, *Physical Review*.

³⁾ See L. VEGARD's researches on the loss of water of the zeolite chabasite, in which the size of the lattice does not change, only the sharpness of the interferences (*Ann. d. Physik* 54, p. 160—164).

⁴⁾ J. R. KATZ, *Die Gesetze der Quellung*, *Kolloidchem. Beihefte* 9, p. 60 and 94 (1916). Röntgenspectrography has however shown since then that these two substances are crystalline, or rather that they contain crystalline elements.

it is found in the ordinary efflorescent salts containing crystal-water.

We were fortunate enough to find one already among these first three substances, in which the elementary body becomes larger on swelling, so that it may now be considered as settled that the water of imbibition in the swelling can just as well go into the lattice as penetrate between micells.

Two other substances which we examined, E. MERCK's amyllum solubile and amyloextrine according to NÄGELI, yielded diagrams that were difficult to explain, the plates for dry and for swollen substance differing too greatly (so that it is uncertain which rings are comparable). Also for these two substances the isotherm was earlier determined as a continuous S-shaped curve.

The substances were examined in the form of powders; only the chitine consisted of leaflets. The powders were examined in tubes of thin nitrocellulose film ¹⁾; it was ascertained that such tubes give no appreciable spectrum of their own. The moist substances had been swollen in liquid water for 24 hours, and then freed from the excess of water between filter paper or on a plate of unglazed porcelain without pressure. The dry substances had been dried at 80° C. in vacuum over sulphuric acid. Of the chitine the rays were transmitted through a leaflet at right angles to its flat surface; it was fixed with wax against the back side of the diaphragm.

The photos were made in the Röntgen cameras described in the preceding paper, which close hermetically; to keep the degree of imbibition constant during the exposure, a dish with phosphorus pentoxide was placed in the camera when dry substances were examined, a quantity of very moist filter paper in the case of moist substances. The moist and the dry substances were photographed successively in the same camera, hence at the same distance from the photographic plate. The nitrocellulose tubes were fastened exactly in the middle of the aperture at the backside of the diaphragm. The spectra were photographed on plates.

As source of light served again a HADDING-SIEGBAHN tube with copper anticathode, which was fed with 15—20 mA. at 50—60 kilovolts; it gives preponderantly monochromatic light ($\lambda = 1,54 \text{ Å.U.}$).

3. *The results of the experiments.*

The inuline was kindly put at our disposal by Prof. H. PRINGS-

¹⁾ These tubes are made by winding thin nitrocellulose film round a steel wire of a thickness of from 1 to $1\frac{1}{2}$ mm., and then gluing it with a mixture of acetone and amylacetate $\bar{a}\bar{a}$.

HEIM (Berlin) for this investigation. It was the purest inuline prepared from dahlia roots. It was a white powder.

In a first experiment we obtained the following values for the diameter of the most important rings in mm. (we shall only be able to give the fainter rings after photometric investigation of the negatives):

| Dry | Moist | Distance preparation plate |
|------------------------------|------------------------------|-------------------------------|
| 23.0 } 23.0 22.9 } | 21.2 } 21.1 21.0 } | 51.5 ± 0.8 |
| 36.0 } 36.0 35.9 } | 33.0 } 32.9 32.8 } | |
| 44.2 } 44.2 44.1 } | 41.0 } 40.9 40.8 } | |

In a second experiment with the same substance the degree of imbibition of the moist substance was now also determined; it was 29 % (parts of water to 100 parts of dry substance); we now found:

| Dry | Moist | Distance preparation plate |
|-------------|-------------|-------------------------------|
| 21.2 | 19.1 | 47.5 ± 0.8 |
| 32.0 | 28.2 | |
| 40.0 | 35.6 | |

The values of the intensest two rings are printed in bold type.

If *all* the water had gone into the crystal lattice, and if this had been isotropically enlarged, the length of the elementary body would have increased by $11\frac{1}{2}\%$ (the specific volume is 0.67, the volume contraction is 0.06 as a maximum). In the first experiment an increase of about 8 % was found in the two darkest rings, in the second experiment of about 10—11 %. *Hence in inuline the imbibition water goes for the greater part into the crystal lattice, and enlarges this in the same degree as the visible dimensions increase (swell).* On the plate the two negatives of the first experiment are reproduced (accurately in the natural size).

The experiments are continued with a view to examining how the water is absorbed by the inuline (stoichiometrically in the crystal

lattice or diffusely distributed through the lattice etc.). We hope to be able to throw some light on questions of this nature in general through the elaborate examination of this particular example.

The *chitine* from lobster's claws had been prepared years before by one of us for his experiments on imbibition. As it is stated there¹⁾ it had been prepared by treatment of the lobster's claws with diluted potassium hydroxide (to remove the albuminous substances), extraction of the calcium carbonate with diluted hydrochloric acid, decoloration with potassium permanganate, and subsequent treatment with sulphurous acid. It consisted of greyish white leaflets, the ash content was only 0.1 %. The substance swollen in water, contained at least 30 % of water. The substance gave a diagram consisting of beautiful rings, the diameter of which was:

| Dry | Moist | Distance preparation plate |
|------|-------|-------------------------------|
| 17.7 | 17.4 | 53.0 ± 0.8 |
| 24.1 | 23.8 | |
| 37.6 | 37.3 | |
| 45.8 | 46.3 | |
| 53.8 | 53.8 | |

The size of the rings is therefore identical within the margin of the errors (on a measurement without photometer).

The *sodium palmitate* of C. F. A. KAHLBAUM was kindly put at our disposal by Prof. H. FREUNDLICH and Mr. JÖRRES. The air-dry substance with 10 % and with 20 % trichlorethylene were examined. The substance gave DEBYE-SCHERRER rings, which though they were not very successful, could yet be measured; their diameter was:

| Airdry | 10 % trichlor- ethylene | 20 % trichlor- ethylene | Distance preparation plate |
|--------|----------------------------|----------------------------|-------------------------------|
| 37.2 | 37.4 | 37.0 | 52.1 ± 0.8 |
| 42.4 | 43.1 | 42.8 | |
| 51.7 | 51.8 | — | |

¹⁾ Gesetze der Quellung, p. 93.

The values show that there is not any difference between the swollen and the not swollen substance. The absorbed quantity is, indeed, not large; but a change of from 3 to 5 %, would yet have been detected with certainty.

The two other substances that were examined, MERCK's *amylum solubile* and amyloextrine (prepared according to NÄGELI)¹⁾ exhibited a secondary complication. The diagrams of the moist substance presented a normal appearance, but those of the dried substances were rougher and less distinct. Possibly the lattice had been partly thrown into disorder through the exit of the water. It is outside the scope of this investigation to enter more fully into this complication. Under these circumstances, however, the conclusions that can be drawn from the negatives, become much less imperative than with the substances discussed above. We will, therefore, only very briefly communicate the results obtained. Of the amyloextrine two rings were measured which looked like single rings in the dry substance, of which, however, the inner one in the moist diagram appeared to consist of at least two rings. These rings (thus measured) had the same diameter in moist and in dry condition. In *amylum solubile* there was, properly speaking only one ring, a dark, narrow one of small diameter, which could be recognized and measured very well in all the diagrams. Its diameter was:

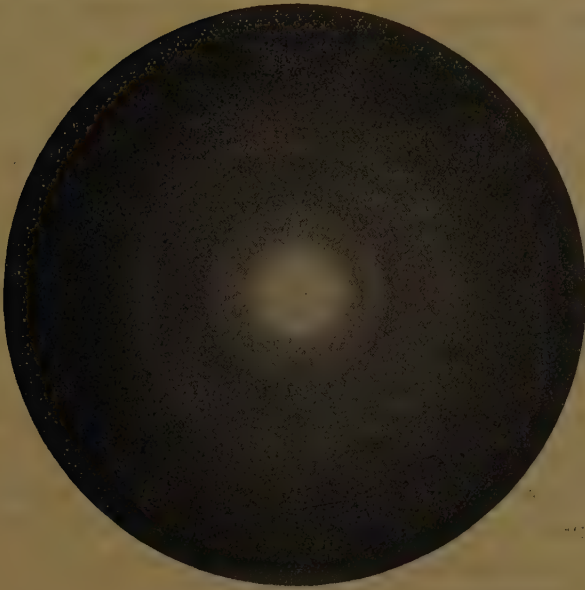
| Dry | | Airdry | Moist | Distance preparation-plate |
|------|------|--------|-------|-------------------------------|
| 33.0 | 32.9 | 31.4 | 29.6 | 53.0 ± 0.8 |

The water-content of the airdry substance was 18 %, that of the moist substance at least 40 %; the specific volume of the dry substance was about 0.6 %. It seems, therefore, that we have a second example in the *amylum solubile* of a substance in which the imbibition-water enters the crystal lattice. But very unfortunately the swelling in this substance is attended with a secondary complication; and under these circumstances the possibility is not entirely excluded, that the results obtained would have to be interpreted in another way.

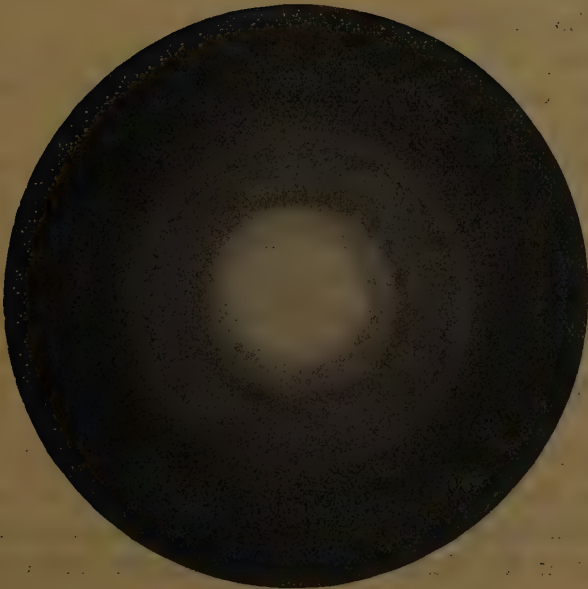
¹⁾ LIEBIG's Ann. d. Chem. 173, p. 218—227.

ROENTGENSPECTRA OF INULINE.

moist.



dry.



(The interference rings have been reproduced as white circles, both represented at their natural size).

CONCLUSIONS.

a. Of the three imbibitive substances mentioned, which give a DEBYE-SCHERRER diagram and are free from secondary complications,
 inuline in water
 chitine in water

sodium palmitate in trichlorethylene,

the first mentioned substance shows an imbibition which goes for by far the greater part into the crystal lattice. A change was found from 8 to 11 %, calculated 11½ %, in the size of the elementary body (of that substance that gives the spectrum).

b. Hence though in cellulose, fibroin, chitine the lattice of the substance giving the spectrum, does not change, when these substances swell in water, there exist other bodies in which the elementary body becomes larger; in inuline this takes place in almost the same degree as in which the visible dimensions increase. Accordingly the "swelling" seems here to be owing to an enlargement of the elementary body. Further experiments were carried out to find more examples of such substances and to see how the lattice varies as a function of the degree of imbibition.

c. The same thing is probably the case with amyllum solubile (whereas of amyldextrine the size of the lattice would not change); but here no certainty was reached, because a secondary complication has a disturbing influence.

d. Accordingly it seems that the imbibition can take place either intermicellarly or intermolecularly; the two cases occur according to the substance that is taken.

e. Our further investigations will, as we hope, clear up the question, which is still entirely unexplained, how such an intermolecular imbibition must be imagined to take place in connection with the structure of the lattice of the crystal. The idea of "intermolecular imbibition" is strictly speaking incompatible with the view still often advocated at present that the whole crystal would really form one single large molecule.

These researches were begun in the Kaiser Wilhelm Institut für Physikalische Chemie und Elektrochemie in Dahlem by one of us (K.); they were completed by both of us in collaboration in the Kaiser Wilhelm Institut für Faserstoffchemie in the autumn of 1923.

Geology. — “Some notes on Foraminifera from the Dutch Indies”.
By L. RUTTEN.

(Communicated at the meeting of June 28, 1924).

1. *On the occurrence of Linderina-limestone in Batjan.*

Some time ago PROF. H. A. BROUWER sent me a lime-sandstone from Batjan, which contained a large number of Linderinae. It is original from the Sg Kubung, to the West of the Sibella-mountains, where it was found as a boulder (n°. 236). In the same river occur folded sandstones and conglomerates, besides marl-limestones with Lithothamnium, and Corals. In all probability all these rocks belong to the Neogene, even though the characteristic fossils are absolutely lacking.¹⁾ It is this very lack of typical older-neogene fossils that renders it plausible to refer the said stones even to the “younger neocene”.

The lime-sandstone is a finely granular rock with many quartz-splinters and numerous small fossil-remains. We find besides the numerous Linderinae, which will be discussed lower down, Amphis-*tegina*?, Operculina, Rotalidae, Globigerinae, very small Nummulites, Textularidae, Nodosaria, Orbulina, Gypsina and scanty Lithothamnium.

Linderina does not abound in the Tertiary depositions of the Malay Archipelago. NEWTON and HOLLAND²⁾ were the first to detect in the old-neogene limestones of North-Borneo remains of this genus established by SCHLUMBERGER³⁾. OSIMO found Linderinae, which she classified as a new species (*L. Paronai*), in (?) eocene limestones from West Celebes⁴⁾, PROVALE described Linderinae from limestones derived from P. Miang near Sangkulirang, East-Borneo⁵⁾, I myself recorded the genus from probably old quaternary strata in the vicinity of Humboldt Bay⁶⁾, and VAN DER VLERK described scanty Linderinae from neogene deposits of Sumbawa⁷⁾.

¹⁾ H. A. BROUWER, Jaarb. Mijnw. Ned. Indië 50. Verh. 2de gedeelte. 1923. p. 71—105.

²⁾ B. NEWTON and R. HOLLAND, Ann. Mag. Nat. Hist. (7), III, 1899, p. 262.

³⁾ C. SCHLUMBERGER, Bull. Soc. géol. de France (3), XXI, 1893, p. 118—120.

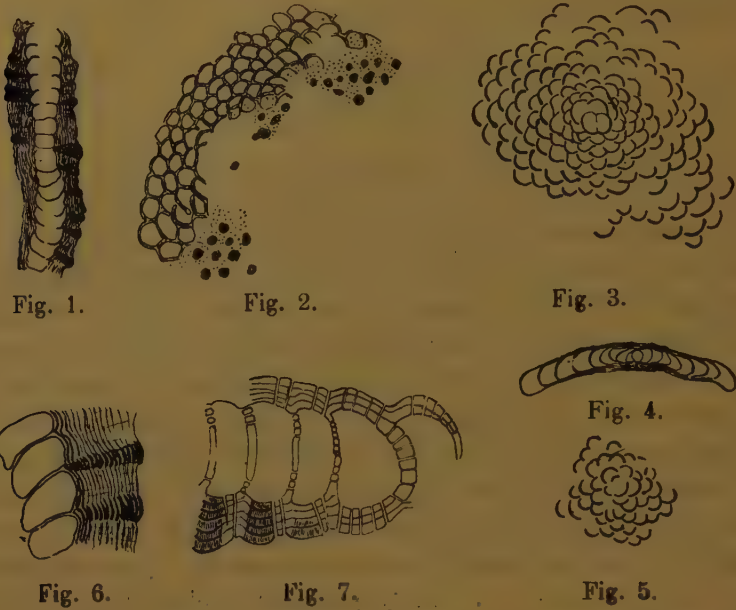
⁴⁾ G. OSIMO, Riv. italiana palaeont. XIV, 1908, p. 38—39.

⁵⁾ I. PROVALE, Riv. italiana palaeont. XV, 1909, p. 88.

⁶⁾ L. RUTTEN, Nova Guinea, VI, 2, 1914, p. 45—46.

⁷⁾ I. VAN DER VLERK, Studiën over Nummulinidae en Alveolinidae etc. Diss. Leiden, 1922.

The fossils now found in Batjan are to be included under *L. Paronai* (figs. 1—7). Their size (rather more than 2 mm.) exceeds that of



Osimo's types; the transverse section (figs. 2, 3) displays the greatest analogy to the forms pictured by *OSIMO* and *PROVALE*, while also the longitudinal sections bear great resemblance to the representations given by the two Italian researchers. Two things, however, are to be noted. In the first place the occurrence of numerous prisms of intermediate skeleton, sometimes resting on the septa of the chamberlets, sometimes, however, rising above the lumen of the chamberlets, (figs. 1, 2, 6, 7). In the second place the occurrence of very wide pores in the vertical, as well as in the horizontal walls of the chamberlets, which are satisfactorily observable when greatly magnified (fig. 7).

As already stated, the genus *Linderina* does not seem to be of great stratigraphical value. This statement may even be emphasized now. To *L. Paronai* belong specimens from the Eocene or the Old-Neogene of Celebes, from young-neogene strata of P. Miang, from very recent caenozoic strata of North-New Guinea (for the fossils formerly classed by me under *Linderina* sp. must no doubt be referred to *L. Paronai*) and from young(?)-neogene strata of Batjan.

2. On the Embryonal Chambers and the median Chamberplane of *Lepidocyclus acuta*.

The descriptions hitherto published of *L. acuta* leave something

to be desired¹⁾. First of all we knew little of the arrangement of the median chambers in the aequatorial plane, a feature that according to VAN DER VLERK (l. c.) is of rather great importance for systematizing *Lepidocyclinae*; secondly, we knew, indeed, that the fossil is microspherical, but what we did not know was the disposal of the first chambers. The truth is that the disposal of the embryonal chambers has been described for only very few microspherical *Lepidocyclinae*, owing to the fact that it is rather difficult to make horizontal sections cut exactly just across these very small chambers. In the whole of CUSHMAN's monograph²⁾ on American *Lepidocyclinae* no satisfactory indications for any form are to be found on the disposal of the first chambers in the microspherical types. As regards Indian *Lepidocyclinae* this arrangement is, to my knowledge known only for a specifically undetermined form from Java, where the first chambers display a distinct spiral arrangement³⁾.

In a median section of *L. acuta* from the Pulu Balang strata near Balik Papan the first chamberlets chanced to be cut through. It then appeared distinctly that they are arranged in a spiral (Fig. 8). This spiral can be traced over one whorl and a half; further on the arrangement of the median chambers becomes irregular and more outwardly it becomes concentric again. It seems, therefore, that the spiral arrangement of the chambers of *Lepidocyclina* does occur occasionally. It is indicative of the close morphological relationship to *Nummulites*.

The same section showed with great precision that the exterior median chambers are disposed in sharply defined pentagons. (Fig. 9).

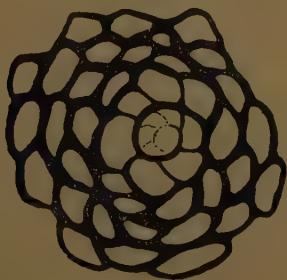


Fig. 8.

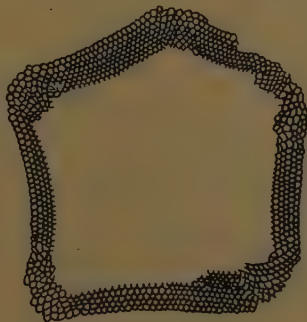


Fig. 9.

¹⁾ L. RUTTEN, These Proceedings 1911, p. 1152—1153. L. RUTTEN, Samml. Geol. Reichsmus. (1), IX, 1914, p. 303—304. I. VAN DER VLERK, Studien over Nummulinidae en Alveolinidae etc. Diss. Leiden, 1922.

²⁾ J. A. CUSHMAN, The American species of *Orthophragmina* and *Lepidocyclina* U. S. Geol. Surv. Prof. Paper 125, D. 1920.

³⁾ R. VERBEEK en R. FENNEMA, Geologie van Java en Madoera. 1896. f. 161, 163.

3. *On the Age of Limestones in the islet of Karama(ng) in the Pare Pare Bay (South Celebes).*

In perusing L. VAN VUUREN's¹⁾ monograph on Celebes it strikes us that writers are at variance among themselves on the age of the limestones occurring in the islet of Kamarang in the Pare Pare Bay. WICHMANN was the first to visit this island and classes the limestones as Neogene; P. and F. SARASIN speak of eocene limestones, while on the authority of DOLLFUS, ABENDANON has determined the rocks in the North of the Pare Pare Bay as Pliocene. VAN VUUREN quotes extensive passages from the literature, to which it suffices only to refer here.

On the basis of these contradictory determinations VAN VUUREN maintains that enclosing the Pare Pare Bay there may be miocene superimposed on eocene rocks; on the other hand, however, he also deemed it possible that Messrs SARASIN had made a mistake.

The original fragments of WICHMANN are present in the Geological Institute of Utrecht, while Dr. A. TOBLER of Basel was so kind as to send me the numbers 257 and 258 of the SARASIN-collection, so that I was in a position to compare the different rocks.

To start with, the limestones collected by WICHMANN and the SARASIN's are strikingly similar: they are yellow to brownish yellow, sometimes reddish-coloured, very porous rocks, in which with the aid of a lens numerous fossils can be perceived. At first sight the rocks look like very young reeflimes, only very little changed diagenetically, such as occur so abundantly in the Archipelago where they are commonly termed "Karang". This first impression is not in any way contravened by WICHMANN's statement that the limestones are only slightly dipping westward. Various preparations made of SARASIN's as well as of WICHMANN's rocks proved their complete identity. The limestones are very porous; the interstitial substance between the pores has been markedly recrystallized; fine biotite platelets and quartz-splinters may be recognized in the rocks as xenoliths. Most fossils in these preparations are well-preserved, very typical Lithothamnium, while there is a nearly equal abundance of small Nummulites, not larger than 2 mm. Beyond these we may identify in much smaller quantity: Rotalidae, Textularidae, Miliolidae, Carpenteria, Nodosaria, Orbulina, Globigerina, ?Gypsina, Lindequina, ?Amphistegina, and ?Polystomella. The presence of the small Nummulites can by no means be adduced in favour of a great age of the formation; they are such as are still living in the Indopacific

¹⁾ L. VAN VUUREN, Het Gouvernement Celebes, 1, 1920, p. 183 e.v.

seas and at any rate are abundantly met with in posteocene depositions. On the other hand the absolute absence of typical littoral Foraminifera of miocene or eocene age in this peculiarly littoral sediment, proves that the *limestones from P. Kamarang belong to the very recent Tertiary and must be classed with the true "raised coralreefs"*, especially when we take into account not only the absence alluded to here, but also the very recent habitus, and the almost horizontal position of the limestones, as well as their occurrence only a few yards above the sealevel¹⁾.

4. *On the occurrence of Lepidocyclina cf. epigona Schubert in globigerina-chert from Skru (West-New-Guinea).*

Of globigerinae-chert — a very peculiar sediment considerably changed diagenetically — from Skru I described an extremely small *Lepidocyclina*²⁾.

I. VAN DER VLERK doubts whether we had really to do here with a *Lepidocyclina*³⁾. This has induced me to carefully re-examine this original rock for the presence of these fossils. One more specimen was recognized, of which a sub-horizontal section could be made. It then became quite evident that the median chambers possess the *Lepidocyclinal* shape. This was proved conclusively during the grinding after the preparation had been accomplished, the shapes of the chambers were not so well-defined, because the section had not been made rigorously horizontal.

Yet my previous description requires some qualification in two respects. First of all the form from Skru was considered as being microspherical. This appears to be wrong, and the mistake was made in consequence of the fact that the previously published transverse section was tangential, not radial, so that the embryonal chambers were not cut. In the new section it is shown clearly that the form is megalospherical and that the first chambers are kidney-shaped. Secondly: the dimensions of the fossils have been underrated, another consequence of the tangential section being mistaken for the median one. The section now proves to be larger, about 2.5 mm. So the fossils from Skru are considerably larger than the type of *L. epigona*, which is only 1 mm. in diameter. Nevertheless I hold that the agreement in features and appearance suffices to justify my opinion that the fossils from Skru are to be classed as *L. epigona*.

¹⁾ See also R. D. M. VERBEEK, *Molukkenverslag H. Mijnw.* 1908. *Wet. Ged.*, p. 61.

²⁾ *Nova Guinea*, VI, 2, 1914, p. 37, 50—51.

³⁾ I. VAN DER VLERK, *l.c.*, p. 26—27.

EXPLANATION OF THE FIGURES.

- Fig. 1, 4. Vertical section through *Linderina Paronai* Osimo. $\times 23$.
 Fig. 2, 3, 5. Horizontal section through *Linderina Paronai* Osimo. $\times 23$.
 Fig. 6, 7. Portions of vertical sections through *Linderina Paronai* Osimo $\times 70$.
 1—6 from Batjan, Sg. Kubung; 7 from Humboldt Bay, North New-Guinea.
 Fig. 8. *Lepidocyclina acuta* Rutten. Embryonal chambers. $\times 350$.
 Fig. 9. *Lepidocyclina acuta* Rutten. Portion of a median horizontal section to
 demonstrate the arrangement of the median chambers. $\times 178$ —9
 from Sg. Blakin, Balikpapan Bay, East Borneo.

Chemistry. — “*Potentiometric Measurement of the Velocity of Diffusion of Metals in Mercury.*” By Prof. ERNST COHEN and Dr. H. R. BRUINS.

(Communicated at the meeting of March 29, 1924).

1. Diffusion is a phenomenon which occurs so frequently in Nature, that on this ground alone considerable importance must be assigned to the exact determination of the coefficient of diffusion.

While we do not wish to take into consideration here physiological and geological processes, it should be noted that measurements of diffusion put us into a position to calculate the resistance which a solute molecule experiences when moving through a liquid under the influence of an external force; so that such measurements can become of great importance as an indication of the molecular condition of liquids.

In a paper recently published ¹⁾, in which we described an accurate method for determining these constants, we have already pointed out that as the result of the great experimental difficulties which are associated with such measurements very significant differences are found (5, 10, and even 60 %) between the results of different authors, even when making measurements on identical objects.

Since we purposed to study not only the influence of the temperature on diffusion phenomena, but also that of pressure, and the method described by us before was not suitable for the latter purpose: we have sought to develop a new experimental method which would put us in a position to measure diffusion-coefficients, within certain limits accurately, at any desired temperature and pressure.

Having attained our object, we will now discuss in detail the procedure, which relates to the determination of the diffusion-coefficients of metals in mercury.

A complete account will appear shortly in the *Zeitschrift für physikalische Chemie*.

2. The principle, which was actually described thirty years ago by DES COUDRES ²⁾, is as follows:

¹⁾ *Zeitschr. f. physik. Chemie* 103, 349 (1923).

²⁾ *Wied. Annalen. N. F.* 52, 191 (1894).

By means of electrolysis of a solution of a salt of the metal under investigation a certain amount of the latter is carried over into the surface of a column of mercury which may be considered as of indefinite length, and is used as cathode; and this metal is allowed to diffuse into the mass of mercury.

The velocity with which this proceeds is measured by determining, at fixed time-intervals after the electrolysis, the potential-difference between the mercury surface (which has now become an extremely dilute amalgam) and a constant electrode which is reversible with respect to the diffusing metal.

This method for measuring diffusion in mercury shows a number of advantages over other methods.¹⁾

We have pointed out previously²⁾ that preference should be given, in general, to those methods of diffusion-measurement in which each experiment gives a large number of values for the coefficient of diffusion, and in which it is unnecessary to disturb the diffusimeter during the whole process.

These conditions were sufficiently accurately fulfilled; in addition our procedure is free from certain disadvantages which are associated with the method, otherwise of a similar type, of G. MEYER³⁾.

3. As will become evident, in testing the new method several systematic errors appeared; however these could be circumvented by controlling certain conditions.

DES COUDRES, who carried out a few preliminary measurements with zinc as the diffusing material, found indeed that the diffusion proceeded in accordance with the theory, but the coefficient calculated by him was a hundred times too small⁴⁾.

Our experiments will show that no reliance can be placed on this result.

4. We have studied in particular the diffusion of cadmium in mercury at 20°.00 C. Since during the diffusion the concentrations were extremely small (at the most 60 mgm. Cd in 100 gm. of mercury, i. e. 0.06 %), the diffusion-coefficient obtained by us can be looked upon as the "true" coefficient⁵⁾ at that temperature, i. e. as that for diffusion at an indefinitely large dilution.

We will first give a description of the apparatus and materials

¹⁾ E.g. VON WOGAU, *Ann. d. Phys.* (4) 23, 345 (1907).

²⁾ *Zeit. f. physik. Chemie* 103, 349 (1923).

³⁾ *Wied. Ann. N. F.* 61, 225 (1897).

⁴⁾ *Verh. der Gesellschaft deutscher Naturforscher und Aerzte. Bremen 1890.* p. 54.

⁵⁾ *Zeit. f. physik. Chemie* 103, 349 (1923), particularly p. 353 and following pp.

used, and of the mode of working; then the results obtained in checking the accuracy of the method, and finally the data for definitely determining the coefficients of diffusion.

5. *The Diffusimeter.* This is illustrated in Fig. 1. It consists of a glass cylinder, the cylindrical form of which was very carefully checked over the portion of its height used in the experiment.

The area of cross-section was previously determined by measuring with the aid of a cathetometer the height of an accurately-weighed mass of mercury introduced into it.

In this vessel is a column of mercury *Hg*; whose height (about 3 cm.) is of an order such that it can be looked upon, for our purposes, as of indefinite length. An aqueous solution of cadmium sulphate is introduced above the mercury.

Am is the constant electrode (reversible in respect to Cd), against which the potential difference of the mercury surface (dilute amalgam) is measured.

Am is a small glass bucket which is filled with an amalgam of Cadmium such that it is heterogeneous at the temperature of the experiment. (The amalgam used by us was one of about 10 parts by weight in a total of 100).

Me is a rod of the purest cadmium (e.g. "KAHLBAUM", impurities about 0.005 %), which is used as anode in the electrolysis.

Both *Am* and *Me* are mounted in the hard rubber stopper *S*, which can be secured in the glass cylinder by means of four ivory screws *I*.

F and *E* are platinum wires permanently connected to *Am* and *Me*. These allow of electrical connection being made between the column of mercury (and the electrode *Me* respectively) and the measuring instruments to be spoken of later. The cadmium electrode *Me* is surrounded by hard filterpaper, which guards against the falling down of small particles of cadmium during the electrolysis.

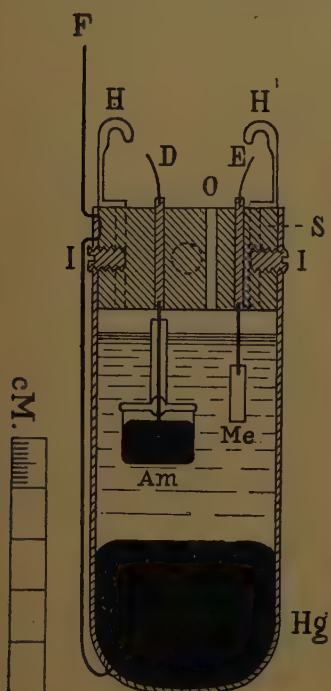


Fig. 1.

6. *The electrical measuring instruments.* The additional apparatus

used in the determination of the diffusion-coefficients, and the connections, are shown diagrammatically in Fig. 2.

The operations to be carried out are as follows:

A. Electrolysis of the cadmium sulphate solution for a known time by a current of known strength. In this *Me* is the anode, *Hg* the cathode. Thus cadmium is deposited electrolytically in the mercury surface at a constant velocity. The current must be constant and is therefore controlled continuously.

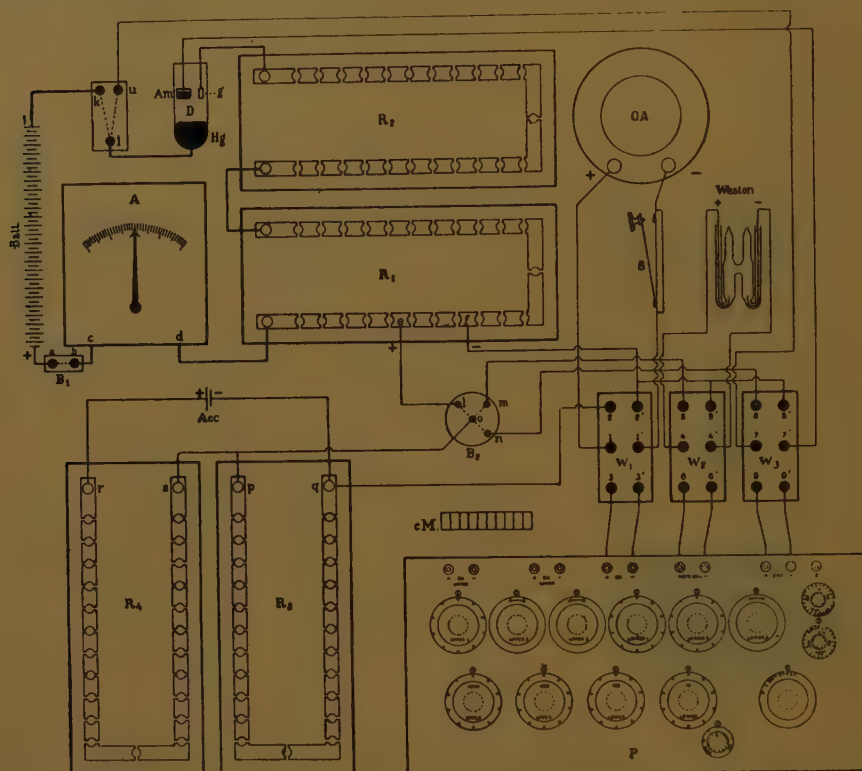


Fig. 2.

B. After the current is stopped, the E.M.F. between *Am* and the surface of the extremely dilute amalgam produced by the electrolysis must be measured at known intervals of time during as long a period as may be desired.

7. (A). In order to carry out the electrolysis in accordance with what was said in paragraph (A), the mercury cup *a* is connected with *b*, and *i* with *k*; the circuit is thus closed for the electrolysing current, which is provided by 25 accumulator cells connected in series.

The current then passes through the milleammeter *A*, several

rheostats R_1 , R_2 , and the diffusimeter g - Hg . The current used was always less than 6 milleamperes.

We used a fairly large battery, while a correspondingly large resistance was connected in the circuit. In this way it was easy to keep the current constant by regulating the resistance. Actually the diffusimeter forms a galvanic element whose E.M.F. is changing continuously.

The determination and continuous control of the current during the electrolysis was carried out by measuring the potential-difference which existed between the points e and f (in the resistance unit R_1) along a known resistance (300—600 ohms), which was included in the circuit.

The potential measurements were carried out according to the method of POGGENDORFF; in this connection were used rheostats R_2 and R_4 , a (DEPREZ-D'ARSONVAL) galvanometer GA which was supported so as to be free from the effects of vibration according to the method of JULIUS¹⁾, GA , and a WESTON cell.

The mercury cups l and o of the switch B , are connected together, while the rocker of the switch W_1 is in the position where its cup 1 is connected with 2, and 1' with 2'. When it is required to switch in the WESTON cell, o is connected with m , whilst the switch-rocker W_2 connects 4 with 5 and 4' with 5'. In order to avoid short-circuits, care is taken that the switch-rocker W_3 , which is not in use at all during the electrolysis, is kept all this time in the position 7—9, 7'—9', or else is entirely removed.

B. Since the E.M.F. between the standard electrode Am and the mercury surface is altering rapidly, particularly just after the interruption of the electrolysis, it is not possible to measure this by the POGGENDORFF method by means of resistance-boxes with plugs. Accordingly we used a WHITE²⁾ potentiometer, which allowed of rapid measurements of E.M.F. (up to 0,1 volt). In this way E.M.F.'s of about 0,005 millivolts could be accurately determined.

In order to measure the E.M.F. during the diffusion, after the interruption of the electrolysing current, the rockers W_1 , W and W_2 were brought into the positions 1—3 (and 1'—3'), 4—6 (4'—6'), and 7—9 (7'—9').

The connection of the potentiometer is shown in Fig. 2. The

¹⁾ Zeitschr. f. Instr. 16, 267 (1896); Wied. Ann. N. F. 56, 151 (1895).

²⁾ The principle of this excellent apparatus is described in the Journ. Amer. Chem. Soc. 36, 1868, 2011 (1914).

Our instrument was the "Single WHITE potentiometer No. 7621" supplied by the firm LEEDS and NORTHRUP in Philadelphia.

terminals "BA upper" and "BA lower" were joined two-by-two to two accumulators connected in parallel. In order to eliminate currents of thermal origin during the measurement the terminals z and E.M.F. of the potentiometer were connected together through a resistance which was equal to that of the liquid in the diffusimeter.¹⁾

8. *Materials. a. Mercury.* Following HULETT's method²⁾, after chemical purification this was twice distilled at a low pressure whilst passing air through it.

b. Cadmium sulphate. This salt (KAHLBAUM, Berlin) was a neutral specimen, such as is used in the construction of WESTON standard cells.

9. *Filling the diffusimeter.* Since the extremely dilute amalgam formed by electrolysis is very easily oxidized by traces of free oxygen,³⁾ the preparation of the liquids and the filling of the diffusimeter were carried out in the following way:

The mercury was heated to 200° for at least a quarter of an hour, and after rapid cooling and filtration poured into the diffusimeter. Then a previously boiled solution of cadmium sulphate was at once poured on top of this. This solution had been cooled in vacuo and filtered as quickly as possible. The stopper S (Fig. 1) was now inserted as rapidly as possible, and through the opening o was poured a certain quantity of oil that had previously been strongly heated and was still warm.

When the apparatus is filled in this way the quantity of free oxygen present is so small that it cannot be detected by the potentiometer in 12 hours.

10. *Temperature regulation.* The diffusimeter is placed in a thick-walled steel cylinder (thickness of wall 5 cm.), which is suspended in a thermostat filled with mineral-oil. (The contents of the thermostat were about 70 litres). The walls of the thermostat, which was built into a wooden box, were lagged with asbestos wool, so that excellent thermal isolation was achieved.

By placing a level on the top of the steel cylinder it was assured that this was horizontal, and that therefore the diffusimeter hung vertically. The top of the steel cylinder lay about 15 cm. below the surface of the mineral-oil in the thermostat; the cylinder was also filled with the oil.

¹⁾ WHITE, Journ. Amer. Chem. Soc. **36**, 1856 (1914); especially p. 1859.

²⁾ Zeitschr. f. physik. Chemie **33**, 611 (1900).

³⁾ HULETT and DE LURY, Journ. Amer. Chem. Soc. **30**, 1805 (1908).

11. In order to prevent the wires *E*, *D* and *F* (Fig. 1), which are connected with the terminals *a*, *b*, *c* (Fig. 3) in the cover of the

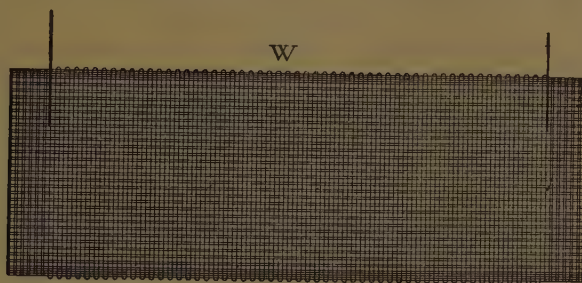


Fig. 3.

steel cylinder, from coming into contact with the wall of the cylinder, the diffusiometer is surrounded with a sheet of celluloid.

12. Since in diffusion experiments any mechanical disturbance must be most carefully avoided, it was not permissible to stir the mineral-oil with an ordinary stirrer. We attained an excellent constancy of temperature in the following way.

We placed a horizontal heating-element in the form of a net (Fig. 3) directly on the bottom of the thermostat, under the base of the steel cylinder (here about 10 cm. thick) and at a distance of about 10 cm. below the latter. This was heated by current from the town electric supply (220 volt D. C.).

An electrical regulator, of a type that we have described before¹⁾, effected the regulation of temperature. The mercury vessel of the regulator stood in the centre of the thermostat, immediately in contact with the wall of the steel cylinder. The temperature variations of the mineral-oil in the thermostat, at the point where the diffusiometer was situated, amounted to 0.03° , when the difference between inside and outside temperature was around 4° ; the variations in the steel cylinder were less than 0.01° .

The diffusion experiments were started only after the diffusiometer had been 12 hours at the temperature of the investigation.

13. *Mechanical disturbances.* In connection with diffusion measurements we described previously²⁾, it had already been noticed that mechanical vibrations produced much less effect when the diffusion took place in a liquid whose upper surface was not free to move. That is the case here, since the diffusiometer is placed in a steel

¹⁾ ERNST COHEN and A. L. TH. MOESVELD, *Zeitschr. f. physik. Chemie* **93**, 385 (1919); especially paragraph 47 and subsequent §§.

²⁾ *Zeitschr. f. physik. Chemie* **103**, 350 (1923); especially § 64.

cylinder completely filled with oil, which is closed at the upper end by a steel cover. Actually any disturbances occurring during the diffusion would become immediately evident in the potentiometric concentration measurements.

The whole apparatus stood on a cement block, which rested on the foundations of the laboratory. No irregularities were ever observed as the result of mechanical vibration.

14. *The Calculation of the Coefficient of Diffusion from the Measurements.* The concentration (c) of the cadmium in mgm. per cc. of amalgam at the surface of the mercury after the time t (measured in seconds), measured from the beginning of the electrolysis, amounts to ¹⁾:

$$c = \frac{2 I h}{q \sqrt{\pi D}} (\sqrt{t} - \sqrt{t - \theta}).$$

Therefore

$$\sqrt{D} = \frac{2 I h}{c q \sqrt{\pi}} (\sqrt{t} - \sqrt{t - \theta}).$$

Here:

I is the current strength in mA.;

h is the electrochemical equivalent of cadmium $\left(0.0005823 \frac{\text{mgr.}}{\text{mA.}}\right)$;

θ the time, in seconds, during which the electrolysis had been carried on;

q the area of the surface of the mercury column in the diffusimeter in square cm.;

D the coefficient of diffusion in $\frac{\text{cm}^2}{\text{sec.}}$.

Here we express the time in seconds, the current strength in milleamperes, and the concentration in mgm. per cc.

15. The area of the surface of the mercury column was taken to be **8.419** square cm., as the mean of 8.422, 8.418, and 8.416.

16. The concentration (c) of the cadmium in the upper surface of the mercury was calculated from the measured e. m. f. (E) between this surface and the electrode Am .

We determined in a direct experimental manner the functional relationship between E and c . This was achieved in the following manner: by means of electrolysis a known mass of cadmium was brought into a weighed quantity of mercury which was placed in

¹⁾ DES COUDRES, Wied. Ann. N. F. **52**, 191 (1894).

the diffusimeter. The precautions against oxidation described in § 9 were applied here. The apparatus was shaken vigorously after the electrolysis, so that the cadmium separated was distributed homogeneously throughout the mercury. Then the e. m. f. between the amalgam and the electrode *AM* was measured (at 20°.00 C.). Similar measurements were repeated with different cadmium concentrations.

Since the NERNST equation for the e. m. f. of a concentration chain applies strictly to this exceedingly dilute solution of cadmium in mercury ¹⁾, the *E*—*c*. curve is given by the equation:

$$E = E_0 - \frac{RT}{nF} \log c.$$

Here *R* is of course the gas-constant (8.312 volt-coulombs), *T* the absolute temperature of the experiment, *n* the valency of the cadmium (*n* = 2), *F* = 96494 coulombs, and *E*₀ is a constant, which is determined in an empirical manner by experiments at different concentrations of cadmium.

Table I gives the corresponding experimental results.

TABLE 1.
Temperature 20°.00 C.

| Weight of Hg in grammes | Time of electrolysis in min. | <i>I</i> in mA. | Weight of cadmium separated in mgr. | Concentration of Cd in mgr. per cc. amalgam | <i>E</i> in Volts | <i>E</i> ₀ in Volts |
|----------------------------|------------------------------------|-----------------|--|--|----------------------|--------------------------------|
| 114.79 | 180 | 10.066 | 63.31 ⁸ | 7.469 | 0.05778 ⁵ | 0.08316 ¹ |
| 114.54 | 140 | 10.056 | 49.19 ⁷ | 5.816 | 0.06093 ⁵ | 0.08315 ³ |
| 114.53 | 105 | 10.041 | 36.84 ⁴ | 4.356 | 0.06459 ⁵ | 0.08316 ⁶ |
| 111.21 | 100 | 10.029 | 35.04 ⁵ | 4.267 | 0.06486 | 0.08317 |
| 111.87 | 210 | 10.029 | 73.59 ⁷ | 8.908 | 0.05555 ⁵ | 0.08315 ⁴ |

As mean value we will take *E*₀ = 0.08316 Volts.

For our equation for *E* we can now write:

$$E = 0.08316 - 0.029058 \log c.$$

17. A reference should be made to the point that, in the calculation of the concentration of the cadmium (in mgm. per cc. amalgam) the equation of HULETT and DE LURY was used, which expresses the density of an extremely dilute amalgam of cadmium as a function of its cadmium content *p*. (number of gm. cadmium per 100 gm. amalgam). This equation:

$$d_{25^\circ} = 13.5340 - 0.0606 p$$

¹⁾ HULETT and DE LURY, Journ. Amer. Chem Soc. **30**, 805 (1908).

gives, allowing for a temperature-coefficient of

$$- (0.0024 + 0.0003 p)$$

a value of

$$13.544 \text{ for } d_{20^{.00}}$$

taking for the concentration of the amalgam a mean value of $p = 0.03$.

18. *Preliminary Experiments.* It was established in preliminary experiments that several factors are present which give rise to certain systematic errors. We will consider here more closely only one of these factors, and that the most notable. This is the influence of the strength of the current during the electrolysis on the (apparent) course of the diffusion as indicated by the measurements. This influence is indicated in the Tables 3, 4, 5, 6, 7, 8.

These contain the data resulting from experiments which were carried out, under otherwise precisely similar conditions, with different current-strengths in the electrolysis.

19. Since in the calculation of D according to the formula given in § 14, the quantity $\log \sqrt{t} - \sqrt{t - \theta}$ plays a part in every experiment (for we always carried out the determination of the e. m. f. after the same interval of time), we give in the next Table 2 this quantity, calculated for different values of $(t - \theta)$. $\theta = 10800$ seconds in all experiments.

TABLE 2.

$$\theta = 10800.$$

| $t - \theta.$ | $\log (\sqrt{t} - \sqrt{t - \theta}),$ |
|---------------|--|
| 600 | 1.91529 |
| 1200 | 1.87451 |
| 1800 | 1.84400 |
| 5400 | 1.73075 |
| 6600 | 1.70472 |
| 7200 | 1.69295 |
| 7800 | 1.68182 |
| 10800 | 1.63392 |
| 18000 | 1.55070 |
| 18900 | 1.54239 |
| 28800 | 1.46669 |

TABLE 3.
 $I = 5.783 \text{ mA.}$

| $t - \theta.$ | $E \text{ (Volt).}$ | $\log c.$ | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|---------------|---------------------|-----------|---|
| 600 | 0.05399 | 1.11487 | 1.355 |
| 1200 | 0.05534 | 1.06841 | 1.388 |
| 1800 | 0.05631 | 1.03502 | 1.409 |
| 6600 | 0.060565 | 0.88861 | 1.457 |
| 18000 | 0.06515 | 0.83083 | 1.482 |
| 21600 | 0.06610 | 0.69815 | 1.488 |

TABLE 4.
 $I = 3.397 \text{ mA.}$

| $t - \theta.$ | $E \text{ (Volt).}$ | $\log c.$ | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|---------------|---------------------|-----------|---|
| 600 | 0.06116 | 0.86813 | 1.456 |
| 1200 | 0.06240 | 0.82546 | 1.469 |
| 1800 | 0.06332 | 0.79381 | 1.476 |
| 6600 | 0.06745 | 0.65175 | 1.496 |
| 10800 | 0.06953 | 0.58019 | 1.501 |
| 18000 | 0.07196 | 0.49649 | 1.504 |
| 28800 | 0.07440 | 0.41260 | 1.503 |

TABLE 5.
 $I = 2.377 \text{ mA.}$

| $t - \theta.$ | $E \text{ (Volt).}$ | $\log c.$ | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|---------------|---------------------|-----------|---|
| 600 | 0.06605 | 0.58882 | 1.548 |
| 1200 | 0.06723 | 0.54821 | 1.547 |
| 1800 | 0.06811 | 0.51785 | 1.546 |
| 6600 | 0.07215 | 0.37890 | 1.543 |
| 7200 | 0.07249 | 0.36710 | 1.543 |
| 18000 | 0.07663 | 0.22481 | 1.540 |
| 28800 | 0.07906 | 0.14101 | 1.542 |

TABLE 6.
 $I = 2.022$ mA.

| $t - \theta.$ | E (Volt). | $\log c.$ | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|---------------|-------------|-----------|---|
| 600 | 0.06821 | 0.62554 | 1.577 |
| 1800 | 0.07022 | 0.55636 | 1.562 |
| 7200 | 0.07458 | 0.40634 | 1.554 |
| 18000 | 0.07870 | 0.26456 | 1.551 |
| 32400 | 0.08114 | 0.18060 | 1.550 |

TABLE 7¹⁾
 $I = 1.132$ mA.

| $t - \theta.$ | E (Volt). | $\log c.$ | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|---------------|-------------|-----------|---|
| 600 | 0.07303 | 0.36874 | 1.613 |
| 1200 | 0.07400 | 0.32719 | 1.618 |
| 1800 | 0.07466 | 0.29849 | 1.605 |
| 5400 | 0.07708 | 0.18929 | 1.575 |
| 6600 | 0.07762 | 0.16397 | 1.571 |
| 18000 | 0.08068 | 0.01177 | 1.557 |
| 21600 | 0.08127 | 0.98029-1 | 1.555 |

TABLE 8.
 $I = 2.497^9$ mA.

| $t - \theta.$ | E (Volt). | $\log c.$ | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|---------------|----------------------|-----------|---|
| 600 | 0.06539 | 0.61153 | 1.540 |
| 1200 | 0.06655 ⁵ | 0.57144 | 1.535 |
| 1800 | 0.06744 | 0.54098 | 1.534 |
| 6600 | 0.07148 | 0.40195 | 1.533 |
| 7200 | 0.07183 ² | 0.38982 | 1.535 |
| 7800 | 0.07215 ⁷ | 0.37863 | 1.527 |
| 18000 | 0.07594 | 0.24847 | 1.529 |
| 18900 | 0.07620 | 0.23952 | 1.534 |
| 28800 | 0.07837 ⁵ | 0.16467 | 1.528 |

¹⁾ In this experiment a small quantity of cadmium was added to the mercury before the commencement of the electrolysis, the concentration of which amounted to 0.3115 mgm. per 10 gm. of mercury. We will explain the reason for this procedure in our following communication. In calculating D this fact is taken into account.

TABLE 9.
 $I = 2.462^6 \text{ mA.}$

| $t - \theta.$ | $E \text{ (Volt).}$ | $\log c.$ | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|---------------|----------------------|-----------|---|
| 600 | 0.06555 ⁵ | 0.60586 | 1.537 |
| 1200 | 0.06671 ⁵ | 0.56498 | 1.537 |
| 1800 | 0.06758 | 0.53617 | 1.525 |
| 6600 | 0.07163 | 0.39679 | 1.526 |
| 7200 | 0.07197 | 0.38509 | 1.525 |
| 7800 | 0.07229 | 0.37408 | 1.525 |
| 18900 | 0.07635 | 0.23436 | 1.527 |
| 28800 | 0.07856 | 0.15830 | 1.528 |

20. Figure 4 gives the value of the coefficient of diffusion as a function of $(t - \theta)$. Each curve refers to a particular diffusion experiment in Tables 3 to 9. Thus the curve I refers to Table 3, II

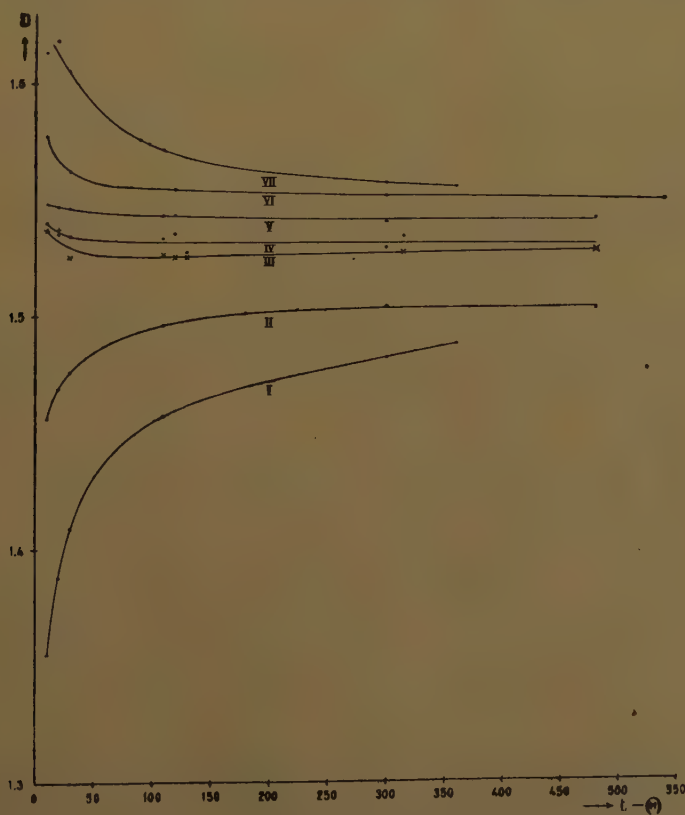


Fig. 4.

to Table 4, V to Table 5, VI to Table 6, VII to Table 7, IV to Table 8, and III to Table 9.

It is apparent from Fig. 4 that:

1. The diffusion coefficient which is calculated from the e. m. f. is *not* independent, in a particular experiment, of the point in time at which the measurement was made. This should be the case according to theory.

2. With larger current strength D is small, and increases with increase of $(t - \theta)$; with smaller current strength D is large and decreases with increase of $(t - \theta)$.

3. Even after fairly long times the same final value of the diffusion coefficient is not reached with different current strengths.

4. The diffusion proceeds according to the theoretical equation only with current strengths which lie within narrow limits.

21. We will give next an explanation of these apparent systematic errors. They are to be ascribed to two different factors. These are:

1. The fact that the upper surface of the mercury in the diffusimeter is not plane, but curved, and therefore greater than the theory implies.

2. The ionic transport that takes place and the accompanying changes in concentration of the cadmium sulphate solution at the mercury surface.

According to the current strength during the electrolysis these two factors have different effects. We will now deal separately with each of the cases mentioned above.

a. The current strength is relatively large during the electrolysis.

Ionic transport takes place during the electrolysis. As the result of this, the concentration of the cadmium sulphate solution decreases where it is in contact with the mercury surface. The layer of more dilute solution produced in this way will be soon forced away by the heavier layer immediately above it, or else will mix with it. But at the edge of the mercury surface, between the glass wall and the mercury surface, there will remain a ring-shaped layer where this forcing-away or mixing will take place with much greater difficulty. Accordingly, in this ring-shaped space the concentration of the cadmium sulphate solution will decrease during the electrolysis. A greater back e.m.f. would therefore arise there, if the concentration of the cadmium in this point of the mercury surface were the same as that in the central part of that surface. Consequently less cadmium is deposited, as the electrolysis proceeds, at the edge of the mercury surface than in the central part, i.e. the cadmium accum-

ulates in the middle portion of the mercury surface. So that although the total mercury surface is actually greater than it would be if it were flat, this peculiar mode of distribution of the cadmium has the same effect as if the surface were smaller than the theory implied. Meanwhile the concentration of the cadmium in the mercury surface will decrease more rapidly after the end of the electrolysis than in the case dealt with by the theory. For as the result of the accumulation of the cadmium that we have just pictured, a steeper gradient of concentration is produced in the central part of the surface.

A calculation which proceeds on the assumption that the surface is *flat* will therefore give, from measurements which are carried out at later periods, coefficients which will show an increase with time.

b. The current strength is relatively small during the electrolysis.

In this case the exhaustion of the solution at the edge of the mercury surface, referred to above, and resulting from ionic migration, will not take place at all, or only to a slight degree; actually it will proceed uniformly at all points in the surface. But since the surface is larger on account of its curvature, than the theory implies, the cadmium concentration will become too small, and accordingly too high a value will be obtained for the coefficient of diffusion.

The deviations from the correct value will here become less with increasing time, since now the fall in concentration at the surface is also smaller than would be the case with a *plane* surface.

Now between the cases indicated by *a* and *b* there must be an interval of corresponding current-strengths over which the distribution of the cadmium during the electrolysis is such that the whole process goes forward as if the surface were not curved. Only those measurements which were carried out with a current-strength which lays within this region can give a diffusion-coefficient agreeing with theory. The experiments which are recorded in the Tables 8 and 9 were carried out approximately within this interval.

From these determinations we find as a mean value:

$$D_{20.00^{\circ}} = 1.528 \times 10^{-5} \frac{\text{cm}^2}{\text{sec.}},$$

a figure which may still be, possibly, a little too high, judging from the shape of the curves in Fig. 3.

22. The correctness of this method of treatment of the problem was confirmed by our investigation on the effect which the concentration of the cadmium sulphate solution, and also a change of

curvature of the mercury surface, produce on the calculated diffusion coefficient. It was established in these that the current strength at which a "gain" in the diffusion-coefficient did not occur is the greater, according as:

1. The concentration of the cadmium sulphate solution is greater, and

2. The curvature of the mercury surface is greater. This result can be deduced as a necessary conclusion from the former discussion; we will show this in our detailed communication.

23. *The Final Measurements.* The results of our final measurements, in connection with which all the precautions described above were applied, are collected in Tables 10—15. It should be noted that the concentration of the cadmium sulphate solution in these experiments was higher than in the experiments of Tables 8 and 9. In accordance with what was said in § 21*a* it was to be expected here that the interval over which D remained constant would only be attained with a fairly high current-strength. This is actually the case, as the tables show. Tables 10—15 give therefore an indication of the reproduceability of the phenomena.

Here the duration (θ) of the electrolysis was also 3 hours (10800 seconds). Measurements were carried out during a period of 8 hours. We deliberately adjusted the current strength (approximately 3.5 mA) so that it was not identical in all experiments (although

TABLE 10.
 $I = 3.673^5 \text{ mA.}$

| $t - \theta.$ | $E(\text{Volt.})$ | $\log c.$ | $D \times 10^6 \frac{\text{cm}^2}{\text{sec.}}$ |
|---------------|----------------------|-----------|---|
| 600 | 0.06044 ⁵ | 0.78170 | 1.521 |
| 1200 | 0.06162 ² | 0.74117 | 1.520 |
| 1800 | 0.06250 ² | 0.71092 | 1.518 |
| 5400 | 0.06582 | 0.59680 | 1.524 |
| 6600 | 0.06656 ⁷ | 0.57099 | 1.522 |
| 7200 | 0.06691 ⁵ | 0.55905 | 1.524 |
| 10800 | 0.06862 ⁷ | 0.50612 | 1.523 |
| 18000 | 0.07104 ⁵ | 0.41692 | 1.523 |
| 18900 | 0.07129 ² | 0.40839 | 1.524 |
| 28800 | 0.07348 | 0.33320 | 1.521 |

naturally it was always constant during the period indicated), in order to avoid an arbitrary factor which might lie in the choice of an identical current strength.

TABLE 11.
 $I = 3.617^8$ mA.

| $t - \theta$. | E (Volt). | $\log c$. | $D \times 10^6 \frac{\text{cm}^2}{\text{sec.}}$ |
|----------------|----------------------|------------|---|
| 600 | 0.06062 | 0.77575 | 1.517 |
| 1200 | 0.06180 | 0.73516 | 1.515 |
| 1800 | 0.06268 ⁵ | 0.70461 | 1.515 |
| 5400 | 0.06600 | 0.59054 | 1.521 |
| 6600 | 0.06676 | 0.56446 | 1.521 |
| 7200 | 0.06709 ⁵ | 0.55285 | 1.520 |
| 7800 | 0.06742 | 0.54175 | 1.520 |
| 10800 | 0.06880 | 0.49410 | 1.519 |
| 18000 | 0.07122 | 0.41097 | 1.518 |
| 18900 | 0.07146 | 0.40272 | 1.518 |
| 28800 | 0.07364 | 0.32760 | 1.513 |

TABLE 12.
 $I = 3.600$ mA.

| $t - \theta$. | E (Volt). | $\log c$. | $D \times 10^6 \frac{\text{cm}^2}{\text{sec.}}$ |
|----------------|----------------------|------------|---|
| 600 | 0.06062 | 0.77568 | 1.503 |
| 1200 | 0.06180 ⁵ | 0.73490 | 1.503 |
| 1800 | 0.06269 | 0.70445 | 1.502 |
| 5400 | 0.06599 ⁷ | 0.59062 | 1.506 |
| 6600 | 0.06677 | 0.56404 | 1.510 |
| 7200 | 0.06711 | 0.55227 | 1.510 |
| 7800 | 0.06742 ⁷ | 0.54141 | 1.508 |
| 10800 | 0.06881 | 0.49383 | 1.506 |
| 18000 | 0.07124 ⁵ | 0.41004 | 1.510 |
| 18900 | 0.07148 ⁵ | 0.40178 | 1.509 |
| 28800 | 0.07370 | 0.32604 | 1.510 |

In deciding on the mean value of D , that figure was always taken which was obtained after the electrolysis had been discontinued for a period of 30 minutes (1800 secs.). This is indicated in the tables by full faced type.

TABLE 13.
 $I = 3.574$ mA.

| $t - \theta$. | E (Volt). | $\log c$. | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|----------------|----------------------|------------|---|
| 600 | 0.06079 | 0.76984 | 1.521 |
| 1200 | 0.06196 | 0.72958 | 1.517 |
| 1800 | 0.06286 | 0.69860 | 1.520 |
| 5400 | 0.06614 | 0.58564 | 1.518 |
| 6600 | 0.06688 ⁵ | 0.56008 | 1.515 |
| 7200 | 0.06722 ⁵ | 0.54838 | 1.515 |
| 7800 | 0.06755 | 0.53728 | 1.515 |
| 10800 | 0.06894 | 0.48944 | 1.514 |
| 18000 | 0.07135 | 0.40642 | 1.513 |
| 18900 | 0.07159 ⁵ | 0.39800 | 1.513 |
| 28800 | 0.07378 | 0.32280 | 1.513 |

TABLE 14.
 $I = 3.3957$ mA.

| $t - \theta$. | E (Volt). | $\log c$. | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|----------------|----------------------|------------|---|
| 600 | 0.06147 ⁵ | 0.74626 | 1.530 |
| 1200 | 0.06264 ⁵ | 0.70601 | 1.527 |
| 1800 | 0.06353 | 0.67554 | 1.526 |
| 5400 | 0.06682 | 0.56224 | 1.526 |
| 7200 | 0.06792 | 0.52446 | 1.526 |
| 7800 | 0.06823 ⁷ | 0.51354 | 1.525 |
| 10800 | 0.06962 ⁷ | 0.46570 | 1.525 |
| 18000 | 0.07204 | 0.38259 | 1.524 |
| 18900 | 0.07279 | 0.37408 | 1.525 |
| 28800 | 0.07448 | 0.29871 | 1.523 |

TABLE 15.
 $I = 3.018^8 \text{ mA.}$

| $t - \theta.$ | $E \text{ (Volt).}$ | $\log c.$ | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|---------------|---------------------|-----------|---|
| 600 | 0.06305 | 0.69205 | 1.545 |
| 1200 | 0.06422 | 0.65180 | 1.541 |
| 1800 | 0.06510 | 0.62151 | 1.540 |
| 6600 | 0.06912 | 0.48317 | 1.533 |
| 7200 | 0.06946 | 0.47147 | 1.530 |
| 7800 | 0.06979 | 0.46069 | 1.530 |
| 10800 | 0.07117 | 0.41262 | 1.532 |
| 18000 | 0.07358 | 0.32968 | 1.530 |
| 18900 | 0.07382 | 0.32142 | 1.529 |
| 28800 | 0.07601 | 0.24606 | 1.527 |

24. The mean values in the final experiments (Tables 10 to 15) are collected in Table 16.

TABLE 16.
 Temperature $20^{\circ}.0 \text{ C.}$

| Current strenght in mA. | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|-------------------------|---|
| 3.673 ⁵ | 1.523 |
| 3.617 ⁸ | 1.518 |
| 3.600 ⁴ | 1.509 |
| 3.574 | 1.515 |
| 3.395 ⁷ | 1.525 |
| 3.018 ⁸ | 1.530 |

Mean of all:

$$D_{20.00^{\circ}} = 1.520 \times 10^{-5} \frac{\text{cm}^2}{\text{sec.}}$$

We found at another concentration of the cadmium sulphate solution the value **1.528** (compare § 21). (less accurate).

25. In conclusion we would offer a remark on the subject of earlier measurements of the velocity of diffusion of cadmium in mercury at 1 atmosphere pressure.

There are available the investigations of G. MEYER¹⁾ and of v. WOGAU²⁾. The technics in these researches (e.g. precautions against oxidation of the extremely dilute amalgam) leaves so much to be desired, that we think we are justified in refraining from discussing their results.

SUMMARY.

A potentiometric method has been described for the determination of the coefficient of diffusion of metals in mercury, which within certain limits can be applied at any desired temperature and pressure.

This method has been used to determine the coefficient of diffusion of cadmium in mercury at 20°.00 C. and 1 atm. pressure.

Utrecht, March 1924.

VAN 'T HOFF-Laboratorium.

¹⁾ Wied. Ann. N. F. **61**, 225 (1897).

²⁾ Ann. d. Phys. (4) **23**, 345 (1907).

Chemistry. — "*The influence of pressure on the velocity of diffusion of metals in mercury.*" By Prof. ERNST COHEN and Dr. H. R. BRUINS.

(Communicated at the meeting of March 29, 1924).

1. Nothing has been known up to the present on the subject of the effect of pressure on the velocity of diffusion of a substance in a liquid.¹⁾

More accurate information as to this effect is not only very desirable in connection with certain geological problems, but also for the solution of definite questions in molecular theory. According to EINSTEIN the following equation holds:

$$D = \frac{RT}{N} B,$$

where D denotes the diffusion-coefficient of a solute, R the gas-constant, T the absolute temperature of the experiment, N AVAGADRO's number, and B the mobility of the solute (i.e. the distance which a molecule covers in unit time in its movement through the liquid, when unit force acts on it).

Hence diffusion measurements offer a direct method for determining the mobility of the molecule. This quantity depends on the molecular forces, which are mutually operative between the molecules; it changes, therefore, according as the condition of the medium, in which the diffusion takes place, is changed. The simplest change which the medium can experience is evidently a compression; the distance between the molecules becomes then smaller, whilst the kinetic energy remains unaltered. Viewed from this standpoint, an investigation of the influence of pressure on the diffusion constant must be looked upon as of special interest.

¹⁾ In their communication on the subject of the effect of high pressure on the properties of solid substances [Zeitschr. f. anorg. Chemie 80, 281 (1913)] JOHNSTON and ADAMS make this statement: "Some investigations have been carried out in this laboratory on the influence of uniform pressure on the velocity of diffusion (solid substances in solid substances). They indicate that uniform pressure increases the velocity of diffusion: but it would be premature to consider this as established, and it is scarcely allowable to draw the conclusion that uniform pressure actually must have such an effect".

2. The limitations are so numerous which the experimental difficulties impose on us in our choice of a suitable system, when undertaking accurate measurements of diffusion at high pressures, that actually only a narrow choice of possibilities remains. We have first to think of a system such that we may be in a position to follow from outside, by electrical or optical methods, the progress of the diffusion which is taking place inside a pressure apparatus.

One might imagine that the method of WEBER ¹⁾ would be applicable to investigations at high pressures which this writer applied to the measurement of the velocity of diffusion of electrolytes in water (involving determinations of e. m. f.). A careful analysis of the difficulties involved in this method brought us to the conclusion not to apply it in this case. Also the systems that he investigated, electrolytes dissolved in water, are of so complicated a nature, and so many factors involved therein alter under pressure (one might mention e. g. the degree of dissociation, the hydration of the diffusing substance, and the polymerisation of the water), that it does not appear at all probable that we shall be in a position, in the near future, to draw any definite conclusions from the pressure-effect observed.

3. The solutions of many metals in mercury are of a much simpler type. The solvent is considered in general to be non-associated, whilst the dissolved metals are present in the monatomic condition.

4. We have applied, for such measurements at high pressures, the potentiometric method for the determination of the velocity of diffusion of metals in mercury described in our previous communication, choosing again cadmium for the diffusing metal.

We carried out the determinations (at 20°.00 C.) at 1 atm. and 1500 atm. pressure. Since preliminary investigations had shown that the change in D produced by a pressure of 1500 atm. was relatively small, it appeared to be superfluous to carry out measurements at intermediate pressures.

Actually the apparent deviations from a linear relationship between p and D are probably too small, considering the degree of accuracy of the measurements, to be determined with any certainty.

5. The determinations at 1500 atm. were carried out in precisely the same manner as at 1 atm. (see our previous paper), but with

¹⁾ Wied. Ann. N.F. 7, 469 (1879); Ook SEITZ, Wied. Ann. N.F. 64, 759 (1898).

this difference, that the diffusimeter in the steel cylinder (the pressure-bomb) (Fig. 3) was subjected to pressure by the introduction of oil.

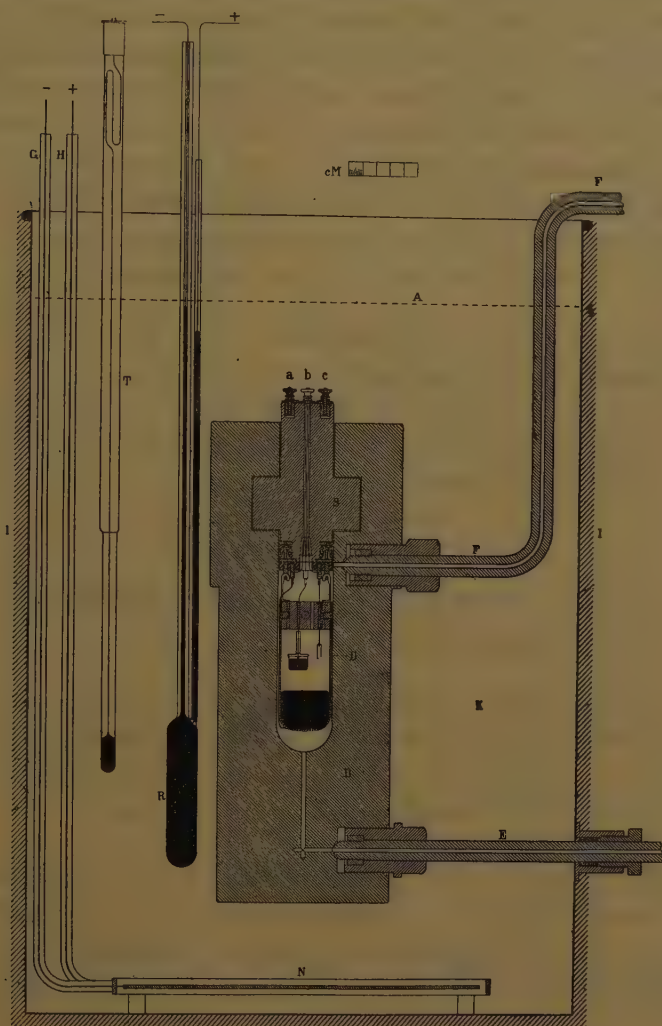


Fig. 1.

We placed in the bomb a mixture of mineral oil with so much vaselin that the whole was fairly viscous at 20° . In this manner the pressure was maintained very constant during the experiment. The use of an automatic pressure device, similar to that which we are accustomed to utilize in our high-pressure investigations¹⁾, is not allowable in diffusion experiments, since any kind of vibration must be avoided.

Through the use of the viscous oil-mixture we were able to

¹⁾ ERNST COHEN and R. B. DE BOER, *Zeitschr. f. physik. Chemie* **84**, 41 (1913).

maintain the pressure so constant for several hours that it was only necessary now and then to regulate it by hand. This was always carried out about an hour before the measurements, whilst the diffusimeter was always brought up to 1500 atm. 12 hours before the beginning of the diffusion, so that, having regard to the high heat of compression, the apparatus should have ample opportunity to assume the temperature (20°.00 C.) of the thermostat. (With regard to temperature-regulation see the previous paper, § 10).

The pressure was read during the experiment on a manometer which was frequently checked by a pressure-gauge¹⁾.

6. With regard to the electrical measuring instruments reference should be made to the previous paper.

7. The diffusion-constant was calculated from the experimental data by means of the equation:

$$\sqrt{D} = \frac{2 I h}{c q \sqrt{\pi}} (\sqrt{t} - \sqrt{t - \theta}).$$

Here the different symbols have the same significance as before (see the former paper, § 14).

8. We used the same diffusimeter at 1 atm. pressure as at 1500 atm. The value of $q = 8.419 \text{ cm}^2$ at 1 atm. (see § 15 of previous paper) must be corrected at 1500 atm. for the compressibility of the glass (2.2×10^{-6}). We obtain $q_{1500 \text{ atm.}} = 8.400 \text{ cm}^2$.

9. It should be further remarked that in the calculation of the cadmium concentration c in mgm. per cc. of the amalgam (see § 16 of our previous communication) account must be taken of the compressibility of the extremely dilute amalgam at the high pressure, which we put equal to that of pure mercury (4×10^{-6}).

We obtain $d_{1500 \text{ atm.}}^{20.00^\circ} = 13.626$.

10. While for the determination of c at 1 atm. pressure we refer to our previous work, we remark that this quantity was also found at 1500 atm. by the measurement of the e.m.f. between the mercury surface and the standard electrode Am (Fig. 1 in the previous paper). We determined the relation between c and the e.m.f.

$$(E)_{1500 \text{ atm.}} = (E_0)_{1500 \text{ atm.}} - \left(\frac{RT}{nF} \log_e c \right)_{1500 \text{ atm.}}$$

¹⁾ ERNST COHEN, KATSUJI INOUE and EUWEN, *Zeitschr. f. physik. Chemie* 75 257 (1910).

in precisely the same way as in the experiments at 1 atm. pressure. As a precautionary measure, the e.m.f. at 1 atm. was determined before every experiment that was carried out at 1500 atm., and after the pressure experiment the e.m.f. at 1 atm. was again determined after temperature equilibrium had been re-established in the diffusimeter.¹⁾

At each pressure the e.m.f. was tested for constancy during a period of several hours. At each measurement the pressure was read.

Table I contains the experimental results for 1 and for 1500 atmospheres respectively.²⁾

TABLE 1.²⁾
Temperature 20°00 C.

| Hg in grammes | Time of electrolysis in min. | I in mA. | Cd deposited in mgm. | Conc. of amalgam mgm. per cc. | $E_{1\text{ atm.}}$ in Volts | $(E_0)_{1\text{ atm.}}$ in Volts | $E_{1500\text{ atm.}}$ in Volts | $(E_0)_{1500\text{ atm.}}$ in Volts |
|---------------|------------------------------|----------|----------------------|-------------------------------|------------------------------|----------------------------------|---------------------------------|-------------------------------------|
| 114.79 | 180 | 10.066 | 63.31 ⁸ | 7.469 | 0.05778 ⁵ | 0.08316 ¹ | 0.05562 | 0.08107 ¹ |
| 114.54 | 140 | 10.056 | 49.197 | 5.816 | 0.06093 ⁵ | 0.08315 ³ | 0.05877 ⁵ | 0.08106 ⁸ |
| 114.53 | 105 | 10.041 | 36.84 ⁴ | 4.356 | 0.06459 ⁵ | 0.68316 ⁶ | 0.06244 | 0.08108 ⁶ |
| 111.21 | 100 | 10.029 | 35.04 ⁵ | 4.267 | 0.06486 | 0.08317 | 0.06269 | 0.08107 ⁹ |
| 111.87 | 210 | 10.029 | 73.597 | 8.908 | 0.05555 ⁵ | 0.08315 ⁴ | 0.05338 | 0.08105 ⁴ |
| | | | | | | Mean: 0.08316 | | Mean: 0.08107 |

Our equations for the calculation of c from the observed e.m.f. are therefore:

$$E_{1\text{ atm.}} = 0.08316 - \left(\frac{RT}{nF} \log c \right)_{1\text{ atm.}} \quad \text{or} \quad E_{1\text{ atm.}} = 0.08316 - 0.029058 \log c_{1\text{ atm.}}$$

and

$$E_{1500\text{ atm.}} = 0.08107 - \left(\frac{RT}{nF} \log c \right)_{1500\text{ atm.}} \\ \text{or} \quad E_{1500\text{ atm.}} = 0.08107 - 0.029058 \log c_{1500\text{ atm.}}$$

11. Before we communicate our results for the final measurements at 1500 atm., we must make reference to the choice of the strength of current used in the electrolysis.

¹⁾ Equilibrium in the amalgam of the standard electrode was also only established after a certain time had elapsed.

²⁾ The values of $E_{1\text{ atm.}}$ and $(E_0)_{1\text{ atm.}}$ will also be found in Table I of our former communication.

It has already been noted that the value of the calculated coefficient of diffusion was not independent of the time during which the process of diffusion was followed after the completion of the electrolysis; also that there exists at 1 atm. pressure a range of values for the current-strength for a given concentration of the cadmium sulphate solution (we used one such that at 1 atm., as well as at 1500 atm., it contained 32 gm. $\text{CdSO}_4 \cdot \frac{8}{9} \text{H}_2\text{O}$ per 100 gm. H_2O) within which the diffusion-constant is found independent of the time.

Meanwhile at 1500 atm. a greater current-strength must be chosen in order to reach this condition than e.g. at 1 atm. pressure. The explanation of this phenomenon is given by the fact that at a high pressure the meniscus of the mercury in the diffusimeter is more sharply curved. Our mode of treatment of the problem in the former paper (§ 21) thus receives fresh support. For the surface-tension of a mercury-water surface increases at a high pressure, as LYNDE¹⁾ had shown.

It can be assumed that this is also the case for the system mercury-cadmium sulphate solution. Accordingly, with increase of pressure the size of the mercury surface will increase.

With a current-strength which will give a constant diffusion-coefficient at 1 atm. pressure, there will be found too high a value

TABLE 2.
 $I = 4.785^6 \text{ mA.}$

| $t - \theta.$ | $E \text{ (Volt).}$ | $\log c.$ | $D \times 10^6 \frac{\text{cm}^2}{\text{sec.}}$ |
|---------------|----------------------|-----------|---|
| 600 | 0.05478 | 0.90482 | 1.437 |
| 1200 | 0.05595 | 0.86447 | 1.434 |
| 1800 | 0.05684 | 0.83391 | 1.435 |
| 5400 | 0.06013 ⁵ | 0.72044 | 1.436 |
| 6600 | 0.06091 | 0.69387 | 1.440 |
| 7200 | 0.06124 | 0.68251 | 1.437 |
| 7800 | 0.06155 ⁵ | 0.67158 | 1.436 |
| 10800 | 0.06295 | 0.62366 | 1.436 |
| 18000 | 0.06536 | 0.54064 | 1.435 |
| 18900 | 0.06559 ⁵ | 0.53255 | 1.433 |
| 28800 | 0.06781 | 0.45640 | 1.436 |

¹⁾ Phys. Rev. **22**, 181 (1906).

for the coefficient at 1500 atm. pressure, when it is determined after the end of the electrolysis, since the concentration of the cadmium at the mercury surface will then be less than e.g. in the experiment at 1 atm. pressure.

The diffusion-coefficients which are calculated from the potential measurements carried out at later points of time will therefore always decrease until a limiting value is reached (vide § 21 of our previous paper). If we are concerned in obtaining at 1500 atm. also such conditions for the diffusion that the deviations may be

TABLE 3.
 $I = 4.751^8$ mA.

| $t - \theta$. | E (Volt). | $\log c$. | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|----------------|----------------------|------------|---|
| 600 | 0.05476 ⁵ | 0.90567 | 1.442 |
| 1200 | 0.05595 | 0.86439 | 1.445 |
| 1800 | 0.05685 | 0.83349 | 1.447 |
| 5400 | 0.06016 | 0.71950 | 1.452 |
| 6600 | 0.06091 ⁷ | 0.69352 | 1.452 |
| 7200 | 0.06126 | 0.68173 | 1.452 |
| 7800 | 0.06158 | 0.67072 | 1.451 |
| 10800 | 0.06297 | 0.62289 | 1.451 |
| 18000 | 0.06539 | 0.53961 | 1.451 |
| 18900 | 0.06563 | 0.53127 | 1.451 |
| 28800 | 0.06783 | 0.45564 | 1.451 |

TABLE 4.
 $I = 4.737^7$ mA.

| $t - \theta$. | E (Volt). | $\log c$. | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|----------------|----------------------|------------|---|
| 600 | 0.05472 | 0.90671 | 1.426 |
| 1200 | 0.05591 | 0.86584 | 1.427 |
| 1800 | 0.05678 ⁷ | 0.83564 | 1.425 |
| 18000 | 0.06534 ⁵ | 0.54115 | 1.432 |
| 18900 | 0.06559 | 0.53272 | 1.433 |
| 28800 | 0.06781 | 0.45640 | 1.431 |

exactly compensated, the current-strength must be taken at a higher figure. Tables 2—7 contain the experimental results for the measurements at 1500 atm.

In the experiment referred to in Table 4 some disturbance

TABLE 5.
 $I = 4.628^9$ mA.

| $t - \theta$. | E (Volt). | $\log c$. | $D \times 10^8 \frac{\text{cm}^2}{\text{sec.}}$ |
|----------------|----------------------|------------|---|
| 600 | 0.05511 | 0.89337 | 1.448 |
| 1200 | 0.05628 ⁵ | 0.85294 | 1.445 |
| 1800 | 0.05719 | 0.82188 | 1.449 |
| 5400 | 0.06050 | 0.70788 | 1.454 |
| 6600 | 0.06125 ⁵ | 0.68190 | 1.454 |
| 7200 | 0.06160 ⁵ | 0.66985 | 1.456 |
| 7800 | 0.06192 ⁷ | 0.65875 | 1.455 |
| 10800 | 0.06331 | 0.61118 | 1.453 |
| 18000 | 0.06573 ⁵ | 0.52773 | 1.455 |
| 18900 | 0.06598 | 0.51930 | 1.455 |
| 28800 | 0.06818 | 0.44359 | 1.456 |

TABLE 6.
 $I = 4.484$ mA.

| $t - \theta$. | E (Volt). | $\log c$. | $D \times 10^8 \frac{\text{cm}^2}{\text{sec.}}$ |
|----------------|----------------------|------------|---|
| 600 | 0.05556 | 0.87781 | 1.459 |
| 1200 | 0.05673 ⁵ | 0.83745 | 1.457 |
| 1800 | 0.05762 | 0.80709 | 1.456 |
| 5400 | 0.06091 ⁷ | 0.69352 | 1.458 |
| 6600 | 0.06167 | 0.66762 | 1.457 |
| 7200 | 0.06201 | 0.65592 | 1.456 |
| 7800 | 0.06234 | 0.64448 | 1.458 |
| 10800 | 0.06373 | 0.59673 | 1.459 |
| 18000 | 0.06614 | 0.51371 | 1.456 |
| 18900 | 0.06638 ⁵ | 0.50537 | 1.456 |
| 28800 | 0.06858 | 0.42991 | 1.455 |

occurred at the start, which became noticeable in the measurement of the e. m. f. Since this vanished later, we nevertheless give the results of the experiment here. As will be evident, this experiment will have only a small effect on the combined result.

TABLE 7.
 $I = 4.4357$ mA.

| $t - \theta$. | E (Volt). | $\log c$. | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
|----------------|----------------------|------------|---|
| 600 | 0.05565 | 0.87472 | 1.448 |
| 1200 | 0.05682 | 0.83460 | 1.444 |
| 1800 | 0.05769 | 0.80458 | 1.441 |
| 5400 | 0.06099 | 0.69102 | 1.443 |
| 6600 | 0.06175 | 0.66488 | 1.444 |
| 7200 | 0.06209 | 0.65318 | 1.443 |
| 7800 | 0.06242 | 0.64190 | 1.444 |
| 10800 | 0.06380 | 0.59432 | 1.442 |
| 18000 | 0.06621 | 0.51139 | 1.440 |
| 18900 | 0.06645 ⁵ | 0.50296 | 1.441 |
| 28800 | 0.06865 | 0.42750 | 1.439 |

12. In Table 8 the results of experiments at 1 atm. (vide our previous paper, Table 16) and at 1500 atm. are collected.

TABLE 8.
Temperature 20° 00 C.

| Pressure 1 Atm. | | Pressure 1500 Atm. | |
|---|---|--|---|
| Current strength in mA. | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ | Current strength in mA. | $D \times 10^5 \frac{\text{cm}^2}{\text{sec.}}$ |
| 3.673 ⁵ | 1.523 | 4.785 ⁶ | 1.436 |
| 3.617 ⁸ | 1.518 | 4.751 ⁸ | 1.451 |
| 3.600 ⁴ | 1.509 | 4.737 ⁷ | 1.432 |
| 3.574 | 1.515 | 4.628 ⁹ | 1.455 |
| 3.395 ⁷ | 1.515 | 4.484 ⁰ | 1.457 |
| 3.018 ⁸ | 1.530 | 4.435 ⁷ | 1.442 |
| Mean of all: | | Mean of all: | |
| $D_{1 \text{ atm.}}^{20^\circ.00} = 1.520 \times 10^{-5} \frac{\text{cm}^2}{\text{sec.}}$ | | $D_{1500 \text{ atm.}}^{20^\circ.00} = 1.446 \times 10^{-5} \frac{\text{cm}^2}{\text{sec.}}$ | |

We find therefore that the coefficient of diffusion of cadmium in mercury at $20^{\circ}.00$ C. decreases with increase of pressure, actually by 5 per cent, for an increase of pressure of 1500 atm.

S U M M A R Y.

The velocity of diffusion of cadmium in mercury at $20^{\circ}.00$ C. and at a pressure of 1500 atm. has been determined according to the method described in the preceding communication.

By an increase of pressure of 1500 atm. this velocity of diffusion is depressed by about 5 per cent.

Utrecht, March 1924.

VAN 'T HOFF *Laboratorium.*

Chemistry. — “*The true specific heat of salt-solutions. Cadmium iodide, ammonium nitrate and zinc sulphate.*” By Prof. ERNST COHEN, Dr. W. D. HELDERMAN and Dr. A. L. TH. MOESVELD.

(Communicated at the meeting of May 3, 1924).

1. In the course of different investigations we came to the conclusion that it was necessary to determine with great accuracy the true specific heat of solutions of cadmium iodide, ammonium nitrate, and zinc sulphate in different concentrations. As the figures relating to it are not to be found in the literature ¹⁾ they are given in this paper.

2. For the determination of the values desired we have used the adiabatic electrical calorimeter described in a former paper by ERNST COHEN and A. L. TH. MOESVELD ²⁾, which enabled us to determine the values in question with an accuracy of 0.5‰. For all the particulars concerning the experimental part and the computations of the observations we refer to this paper.

A. Cadmium iodide solutions. (Temp. 19° C.)

3. The necessary solutions were prepared by diluting a concentrated solution. The CdJ₂ used, was obtained from pure cadmium (so-called “Kadmium KAHLBAUM”, impurities 0.005%) and pure iodine, as described in a former paper ³⁾. The concentration of the solutions obtained in this manner was controlled by determining the specific volume at 30°.00 C. This was possible by using the equation formerly found ⁴⁾:

$$(v_c)_{30^{\circ}.00} = 1.00435 - 0.802924 c + 0.016857 c^2,$$

which gives the specific volume at that temperature as a function of the concentration (grams CdJ₂ per 100 grammes of solution).

¹⁾ The figures mentioned in the literature, referring to the salts investigated by us always indicate the *mean* specific heats within wide temperature intervals. As these intervals in our determinations were not greater than 5 degrees, we may consider the figures found as *true* specific heats.

²⁾ These Proceedings 28, 883 (1919–1920); Zeitschr. f. physik. Chemie 95, 305 (1920); 100, 151 (1922).

³⁾ ERNST COHEN and H. R. BRUINS, Zeitschr. f. physik. Chemie 95, 43 (1918).

⁴⁾ ERNST COHEN, C. W. G. HETTERSCHIJ and A. L. TH. MOESVELD, Zeitschr. f. physik. Chemie 94, 210 (1920).

4. Table I contains the results obtained. All the weighings were reduced to vacuum.

TABLE I.
True Specific Heat of Cadmium iodide solutions of different concentrations.
Temp. 19° C.

| Weight of solution in gr. | (v_2)30°00 | Concentration gr. of salt in 100 gr. of solution | Average deviation of the temp. of the calorimeter from the surroundings | Correction for radiation | Specific heat of the solution found | Spec. heat of the solution found average | Spec. heat calculated (1) | Spec. heat calculated (1) — found $\times 10^4$ | Spec. heat calculated (1) | Spec. heat calculated (2) — found $\times 10^4$ |
|---------------------------|----------------|--|---|--------------------------|-------------------------------------|--|---------------------------|---|---------------------------|---|
| 850.0 (water) | — | — | —0.025 | +0.0035 | water equivalent 36.52 | water-equivalent 36.40 | (0.9992) | — | — | — |
| 850.0 | — | — | —0.017 | +0.0025 | 36.40 | | | | | |
| 850.0 | — | — | —0.010 | +0.0014 | 36.31 | | | | | |
| 1320.0 | 0.64394 | 44.30 | —0.004 | +0.001 | 0.5879 | 0.5877 | 0.5868 | —9 | 0.5877 | ± 0 |
| 1320.0 | 0.64394 | 44.30 | —0.005 | +0.001 | 0.5875 | | | | | |
| 1190.0 | 0.71587 | 35.41 | —0.009 | +0.001 | 0.6697 | 0.6697 | 0.6715 | +18 | 0.6696 | —1 |
| 1190.0 | 0.71587 | 35.41 | +0.005 | —0.001 | 0.6696 | | | | | |
| 1103.0 | 0.77411 | 28.21 | +0.012 | —0.002 | 0.7374 | 0.7378 | 0.7394 | +16 | 0.7380 | +2 |
| 1103.0 | 0.77411 | 28.21 | —0.002 | 0.000 | 0.7382 | | | | | |
| 986.0 | 0.86194 | 17.41 | +0.053 | —0.008 | 0.8407 | 0.8410 | 0.8401 | —9 | 0.8408 | —2 |
| 986.0 | 0.86194 | 17.41 | —0.006 | +0.001 | 0.8413 | | | | | |
| 910.0 | 0.93527 | 8.43 | 0.000 | 0.000 | 0.9247 | 0.9250 | 0.9925 | —25 | 0.9248 | —2 |
| 910.0 | 0.93527 | 8.43 | —0.004 | +0.001 | 0.9253 | | | | | |

5. From the values in the fourth and eighth columns we have calculated by the method of least squares an equation of the second degree, which shows the relation between concentration and specific heat:

$$w_c = 0.9992 - 0.009033 c - 0.00000625 c^2 \quad (1)$$

It gives however differences which are greater than the experimental errors. The equation of the third degree:

$$w_c = 0.9992 - 0.0085097 c - 0.0000424 c^2 + 0.00000056 c^3 \quad (2)$$

on the contrary corresponds exactly with the results, as is seen from columns 8 and 12 in table I.

B. Solutions of Ammonium nitrate (Temp. 32° 3 C.).

6. As the investigations in which the specific heat of these solutions plays a part had to be made at 32° 3 C. (transition point of the modification III into IV), the specific heats of the solutions were also determined at this temperature ¹⁾.

7. We started with a preparation which had been obtained by crystallizing an already rather pure preparation from water. After crystallization impurities could not be found in 3 gr. of the salt. The salt was completely dry, as was proved by duplicate determinations of the density at 32° 3 C., which are in complete agreement with earlier investigations ¹⁾.

8. First of all we determined the curve which represents the specific volume of ammonium nitrate solutions as a function of the concentration (at 32° 3 C.). For this purpose the specific volume was determined of solutions which had been prepared by weighing in salt and water. The pycnometer we used was an instrument as devised by SPRENGEL-OSTWALD-EYKMAN ²⁾.

Table II contains the results.

9. By means of the method of least squares we find from these observations:

$$(v_c)_{32.03} = 1.00507 - 0.00392087 c + \\ + 0.0000051003 c^2 - 0.00000000566 c^3 \quad (a)$$

Table 2, columns 5, 6 and 7, shows the agreement between the values calculated and the values found for v_c .

¹⁾ ERNST COHEN and J. KOOP, These Proceedings 32, 943 (1924); Zeitschr. f. physik. Chemie 109, 81 (1924).

²⁾ Recueil des Trav. Chim. des Pays-Bas 13, 24 (1894).

TABLE II.

Specific Volume of solutions of Ammonium nitrate of different concentrations.

Temp. 32° 3 C.

| Weight of salt in gr. | Weight of solution in gr. | Concentration gr. of salt in 100 gr. solution | v_c found | v_c average found | v_c calculated | v_c calculated—found $\times 10^5$ |
|-----------------------|---------------------------|---|-------------|---------------------|------------------|--------------------------------------|
| 3.8536 | 38.4570 | 10.01 | 0.96628 | 0.96628 | 0.96633 | + 5 |
| 3.8536 | 38.4570 | 10.01 | 0.96627 | | | |
| 4.8339 | 24.1098 | 20.05 | 0.92838 | 0.92839 | 0.92844 | + 5 |
| 4.8339 | 24.1098 | 20.05 | 0.92839 | | | |
| 7.6877 | 25.5763 | 30.08 | 0.89164 | 0.89165 | 0.89158 | — 7 |
| 7.6877 | 25.5763 | 30.08 | 0.89165 | | | |
| 9.7456 | 24.3950 | 39.95 | 0.85628 | 0.85630 | 0.85621 | — 9 |
| 9.7456 | 24.3950 | 39.95 | 0.85631 | | | |
| 11.8383 | 23.7670 | 49.81 | 0.82166 | 0.82167 | 0.82173 | + 6 |
| 11.8383 | 23.7670 | 49.81 | 0.82167 | | | |
| 14.2543 | 23.9459 | 59.52 | 0.78848 | 0.78850 | 0.78855 | + 5 |
| 14.2543 | 23.9459 | 59.52 | 0.78852 | | | |
| 15.7044 | 22.5557 | 69.62 | 0.75494 | 0.75492 | 0.75489 | — 3 |
| 15.7044 | 22.5557 | 69.62 | 0.75489 | | | |

10. The concentration of the solutions whose specific heat was measured, was obtained with the equation (a) by determining the specific volume at 32° 3 C. and calculating c .

11. In table III we give the results of the calorimetric determinations.

12. By the method of least squares we find for the relation between the specific heat (w_c) and the concentration (c) of the solutions:

$$(w_c)_{32.3} = 0.9985 - 0.00765857 c + \\ + 0.000030811 c^2 - 0.00000006473 c^3.$$

TABLE III.

True Specific Heat of solutions of Ammonium nitrate of different concentrations.
Temp. 32°.3.

| Weight of solution in grammes | Specific volume at 32°.3 v_c | Concentration gr. salt in 100 gr. of solution | Average deviation of the temp. of the surroundings in C°. | Correction for radiation C°. | Specific heat of the solution found | Specific heat of the solution found average | Specific heat of the solution calculated | Specific heat calculated — found $\times 10^4$ |
|-------------------------------|--------------------------------|---|---|------------------------------|-------------------------------------|---|--|--|
| 1133.09 | 0.75548 | 69.46 | —0.022 | +0.003 | 0.5931 | 0.5935 | 0.5935 | 0 |
| 1131.31 | 0.75548 | 69.46 | —0.009 | +0.001 | 0.5939 | | | |
| 1076.14 | 0.78257 | 61.31 | +0.018 | —0.003 | 0.6298 | 0.6297 | 0.6298 | + 1 |
| 1078.33 | 0.78257 | 61.31 | +0.011 | —0.002 | 0.6295 | | | |
| 1036.96 | 0.82088 | 50.03 | +0.041 | —0.006 | 0.6844 | 0.6847 | 0.6844 | — 3 |
| 1038.00 | 0.82088 | 50.03 | +0.015 | —0.002 | 0.6850 | | | |
| 996.10 | 0.85589 | 40.06 | —0.009 | +0.001 | 0.7376 | 0.7379 | 0.7370 | — 9 |
| 996.58 | 0.85589 | 40.06 | +0.053 | —0.007 | 0.7381 | | | |
| 957.57 | 0.89175 | 30.03 | —0.025 | +0.003 | 0.7932 | 0.7931 | 0.7945 | +14 |
| 958.13 | 0.89175 | 30.03 | +0.001 | +0.000 | 0.7930 | | | |
| 915.99 | 0.93155 | 19.22 | —0.010 | +0.001 | 0.8622 | 0.8620 | 0.8622 | + 2 |
| 914.89 | 0.93155 | 19.22 | —0.015 | +0.002 | 0.8618 | | | |
| 868.25 | 0.96624 | 10.03 | —0.002 | 0.000 | 0.9250 | 0.9254 | 0.9247 | — 7 |
| 871.65 | 0.96624 | 10.03 | —0.004 | 0.000 | 0.9258 | | | |

From the columns 6, 7 and 8 of tabel III we see the good agreement between the values given by the equation and the results of the measurements.

C. Zinc sulphate solutions. (Temp. 19° C.).

13. The salt was prepared by crystallizing an already rather pure preparation. By weighing we made a solution of 34.21 % (gr. ZnSO_4 in 100 gr. of solution) and controlled the composition by gravimetric analysis, according to the method of EULER¹⁾. From this

¹⁾ Zeitschr. f. anorg. Chemie 25, 146 (1900).

standard solution were prepared others, whose concentration was $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ and $\frac{1}{16}$ of the original solution, the composition of the solutions $\frac{1}{2}$ and $\frac{1}{4}$ being controlled again by analysis. The specific volumes were determined too (at $30^{\circ}.00$ C.). They have also been inserted in table IV which contains the results of the determinations of the specific heat.

TABLE IV.

True Specific Heat of Zinc sulphate solutions of different concentrations.

Temp. 19° C.

| Concentration gr. of salt in 100 gr. of solution | Temperature rise in degrees C. | Specific heat of solution found | Specific heat of solution found average | Specific heat of solution calculated | Specific heat calculated— found $\times 10^4$ | Specific volume of solution at $30^{\circ}.00$ C. |
|---|--------------------------------------|---------------------------------------|---|--|---|--|
| 0 | 5.148 | water equivalent 43.92 | | | | |
| 0 | 5.036 | " 44.06 | — | — | — | 1.00435 |
| 0 | 5.144 | " 43.98 | | | | |
| 34.21 | 5.139 | 0.6692 | | | | |
| 34.21 | 5.166 | 0.6683 | 0.6688 | 0.6692 | + 4 | 0.68960 |
| 25.66 | 5.148 | 0.7352 | | | | |
| 25.66 | 5.146 | 0.7356 | 0.7354 | 0.7346 | — 8 | 0.76222 |
| 17.14 | 5.156 | 0.8112 | | | | |
| 17.14 | 5.148 | 0.8115 | 0.8113 | 0.8113 | ± 0 | 0.83908 |
| 8.60 | 5.143 | 0.8989 | | | | |
| 8.60 | 5.136 | 0.8989 | 0.8989 | 0.8992 | + 3 | 0.91958 |
| 4.30 | 5.148 | 0.9472 | | | | |
| 4.30 | 5.145 | 0.9476 | 0.9474 | 0.9478 | + 4 | 0.96124 |
| 2.15 | 5.156 | 0.9725 | | | | |
| 2.15 | 5.150 | 0.9723 | 0.9724 | 0.9731 | + 7 | 0.98258 |
| 0 | 5.147 | water equivalent 44.08 | — | — | — | 1.00435 |

14. By means of the method of least squares we obtained from the values found for the specific heat:

$$w_c = 0.9992 - 1.290 c + 0.773 c^2$$

(sum $c = 1$).

From the columns 5, 6 and 7 in table IV we find that there is good agreement between this equation and the experimental results.

S U M M A R Y.

The true specific heat of solutions of different concentrations of cadmium iodide (at 19° C.), ammonium nitrate (at 32° 3 C.), and zinc sulphate (at 19° C.) was determined in an electrical adiabatic calorimeter, with an accuracy of 0.5 ‰.

Utrecht, April 1924.

VAN 'T HOFF-*Laboratorium*.

Physics. — “*Termnames of some Multiplets in the Ironspectrum*” ¹⁾.
By S. GOUDSMIT. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of February 23, 1924).

F. M. WALTERS ²⁾ succeeded to find so called “multiplets” ³⁾ in the complicated ironspectrum. These multiplets show the same structure as the combination between two threefold spectral-terms, e. g. the *p-d* combinations in the spectra of the alkaline earths. Such a group of lines is called multiplet if one of the combining terms is more than threefold.

The structure of a multiplet is governed by the restriction-rules for the “inner” quantumnumbers *j*, which were first used by SOMMERFELD ⁴⁾. The relations between these quantumnumbers and the azimuthal quantumnumbers *l*, were found by LANDÉ ⁵⁾, who also got formulae for the ZEEMANeffects of the spectrallines.

LANDÉ uses the following quantumnumbers:

1st. The azimuthal quantumnumber *K*, which gives the moment of momentum of the emitting electron. $K = \frac{1}{2}$ for the *s*-terms, $\frac{3}{2}$ for the *p*-terms, $\frac{5}{2}$ for the *d*-terms, etc.

2nd. The quantum *R*, which gives the moment of momentum of the atom without the emitting electron. $R = \frac{1}{2}$ for all the spectral-

¹⁾ Since the original publication, several articles have been published on the same subject:

O. LAPORTE, ZS. f. Phys. 23, 135 and 26, 1, 1924.

M. A. CATALÁN, Anal. Fis. y. Quim. XXI, 464.

H. GIESELER u. W. GROTRIAN, ZS. f. Phys. 22, 245, 1924.

O. LAPORTE's articles are by far the completest, for the rest the absorption experiments of H. GIESELER and W. GROTRIAN are of great importance.

²⁾ F. M. WALTERS JR. Journ. Wash. Ac. of Sc. 13, 1923, p. 242.

³⁾ M. A. CATALÁN, Phil. Trans A 223, 1922, p. 125.

H. GIESELER, Ann. der Phys. 69, 1922, p. 147.

⁴⁾ A. SOMMERFELD, Atombau u. Spektrallinien 3. Aufl., p. 446.

⁵⁾ A. LANDÉ, ZS. f. Phys. 15, p. 189.

terms of a singlet-term-system, $\frac{2}{2}$ for a doublet-system, $\frac{3}{2}$ for a triplet-system, etc.

3rd. The quantumnumbers J for a complex-term, which is determined by R and K , are given by the formula

$$|K-R| + \frac{1}{2} \leq J \leq |K+R| - \frac{1}{2} \quad . \quad . \quad . \quad (1)$$

Using the rules of SOMMERFELD and LANDÉ, it is not difficult to determine the values of R , K and J for the terms, which form the iron-multiplets found by WALTERS; this means, that it is possible to determine the termsymbols and the term-system.

The following scheme contains the frequencies $\nu_{vac.}$ and the term-differences of an iron-multiplet, called provisionally XY . Each of the terms X and Y has a certain quantumnumber J . The fact that not all the 25 combinations XY are appearing, is easily explained by the restriction-rule for J , being:

$$J \rightarrow \begin{matrix} J+1. \\ J \\ J-1. \end{matrix}$$

| J | X_1 415.9 (7) $4\frac{1}{2}$ | X_2 288.1 (6) $3\frac{1}{2}$ | X_3 154.1 (5) $2\frac{1}{2}$ | X_4 89.9 (4) $1\frac{1}{2}$ | X_5 (3) $\frac{1}{2}$ |
|-----------------------------------|-----------------------------------|-----------------------------------|-----------------------------------|----------------------------------|----------------------------|
| Y_1 (7) $4\frac{1}{2}$ 240.2 | 25900.00 | 25484.03 | | | |
| Y_2 (6) $3\frac{1}{2}$ 199.5 | 26140.19 | 25724.24 | 25436.14 | | |
| Y_3 (5) $2\frac{1}{2}$ 139.7 | | 25923.77 | 25635.67 | 25451.45 | |
| Y_4 (4) $1\frac{1}{2}$ 71.7 | | | 25775.35 | 25591.23 | 25501.3 |
| Y_5 (3) $\frac{1}{2}$ | | | | 25662.35 | |

If we give to the X -terms successive quantumnumbers J , chosen quite arbitrarily, increasing, according to LANDÉ's interval-rule, in the direction of the growing term-differences, then by the restriction-rule, these numbers are also determined for the Y -terms.

Suppose e.g. the terms X_1, X_2 , etc. have respectively $J=7, 6, 5, 4, 3$, then the terms Y_1, Y_2 , etc. will necessarily get the quantumnumbers $J=7, 6, 5, 4, 3$, in order to explain the observed combinations. But than the missing of the combination $X_4 Y_5$ remains still unexplained.

SOMMERFELD and LANDÉ however gave the condition, that the combination

$$J = \frac{1}{2} \rightarrow J = \frac{1}{2}$$

is not allowed, as an addition to the restriction-rule for J . Obviously the combination $X_4 Y_5$ must represent such a case. Thus both X_4 and Y_5 must get the value $J = \frac{1}{2}$. This result enables us to calculate the absolute values of J , being:

$$J = 4\frac{1}{2} \text{ for } X_1, 3\frac{1}{2} \text{ for } X_2, 2\frac{1}{2} \text{ for } X_3, 1\frac{1}{2} \text{ for } X_4 \text{ and } \frac{1}{2} \text{ for } X_5.$$

The Y -terms have the same quantumnumbers.

Formula (1) gives us R and K :

$$R = 2\frac{1}{2} \quad K = 2\frac{1}{2}.$$

Consequently we have here the case of two d -terms of a quintet-system.

Owing to the structure of formula (1), R and K are not always given unambiguously. In such a case it will be sufficient to calculate the ZEEMANeffect for a single line of the multiplet by the rules of LANDÉ and to compare with the experiments. On the other hand the ZEEMANeffects form a confirmation for the whole.

As the same termdifferences, in other words the same complex-terms, appear several times in these ironmultiplets, it is quite possible to derive from the above mentioned multiplet the values of J for all other terms.

It resulted that the 20 iron-multiplets are combinations of 13 different multiple terms of a triplet- and a quintet-termsystem. Two of them are irregular (x and y), probably the connected multiplets were not completely observed¹⁾.

Some new combination-multiplets of these terms could be found and also a new term²⁾.

Table I contains the termsymbols and the termdifferences.

Table II compares the calculated ZEEMANeffects with those indicated by F. M. WALTERS, according to observations of A. S. KING³⁾.

Table III shows the observed combinations and

¹⁾ According to the ZEEMANeffect of multiplet (12) the x -term is probably partially a D -term.

²⁾ The multiplets 1—20 are found by F. M. WALTERS, the others are new.

³⁾ A. S. KING, Contr. Mount Wilson, Vol. III, p. 82.

Table IV the frequencies of the multiplets in the usual schematic form.

It deserves attention, that the multiplet-lines given here, coincide, almost totally with the "low-temperature lines" given by A. S. KING ¹⁾, in so far as they are comprised in the spectral-region examined by him.

TABLE I.

| Termname | | | Termdifferences | | | | | |
|----------------|------------------|---------|--------------------|----------------|----------------|----------------|----------------|---------------|
| | <i>k</i> | | $J = 5\frac{1}{2}$ | $4\frac{1}{2}$ | $3\frac{1}{2}$ | $2\frac{1}{2}$ | $1\frac{1}{2}$ | $\frac{1}{2}$ |
| Tripletssystem | d' ($P'' ?$) | 3 (2 ?) | | | 390.6 | 252.0 | | |
| | d'' ($P' ?$) | 3 (2 ?) | | | 412.9 | 263.8 | | |
| | f' | 4 | | 584.7 | 407.6 | | | |
| | f'' | 4 | | 476.5 | 358.5 | | | |
| | g' | 5 | 388.4 | 311.8 | | | | |
| Quintetssystem | D' | 3 | | 415.9 | 288.1 | 184.1 | 89.9 | |
| | D'' | 3 | | 344.0 | 261.5 | 173.2 | 86.6 | |
| | D''' | 3 | | 240.2 | 199.5 | 139.7 | 71.1 | |
| | D'''' | 3 | | 384.3 | 272.6 | 175.2 | 86.0 | |
| | F' | 4 | 448.5 | 351.3 | 257.8 | 168.9 | | |
| | F'' | 4 | 344.1 | 289.2 | 218.4 | 144.9 | | |
| | F''' | 4 | 292.3 | 227.9 | 164.9 | 106.8 | | |
| | y | ? | 474.9 | 354.3 | 244.8 | | | |
| | | | $J = 4\frac{1}{2}$ | $3\frac{1}{2}$ | $2\frac{1}{2}$ | $2\frac{1}{2}$ | $1\frac{1}{2}$ | |
| | x | (3 ?) | 411.2 | 294.4 | 145.4 | 70.2 | | |

¹⁾ A. S. KING, Contr. Mount Wilson, Vol. VI, p. 351.

TABLE II.

| Combination | | ZEEMAN effect | | |
|----------------|----------------|--|---|-------------------------------------|
| J | J | Calculated ¹⁾ | Observed (multiplet number) | |
| f | f | | 18 | |
| $4\frac{1}{2}$ | $4\frac{1}{2}$ | $\frac{(0)5}{4}$ | $\frac{(0)6}{5}$ | |
| $3\frac{1}{2}$ | $3\frac{1}{2}$ | $\frac{(0)13}{12}$ | $\frac{(0)12}{11}$ | |
| $2\frac{1}{2}$ | $2\frac{1}{2}$ | $\frac{(0)2}{3}$ | $\frac{(0)2}{3}$ | |
| f | g | | 19 | |
| $4\frac{1}{2}$ | $5\frac{1}{2}$ | $\frac{(\overline{0} \ 1 \ 2 \ 3 \ 4) \ \overline{20} \ 21 \ 22 \ 23 \ 24 \ 25 \ 26 \ 27 \ 28}{20}$ | $\frac{(0)5}{4}$ | |
| $3\frac{1}{2}$ | $4\frac{1}{2}$ | $\frac{(\overline{0} \ 2 \ 4 \ 6) \ \overline{57} \ 59 \ 61 \ 63 \ 65 \ 67 \ 69}{60}$ | $\frac{(0)7}{6}$ | |
| $2\frac{1}{2}$ | $3\frac{1}{2}$ | $\frac{(\overline{0} \ 1 \ 2) \ \overline{7} \ 8 \ 9 \ 10 \ \overline{11}}{12}$ | $\frac{(0)7}{8}$ | |
| D | D | Every line $\frac{(0)3}{2}$ | $\frac{3}{2} \ \frac{16}{2}$ Every line $\frac{(0)3}{2} \ \frac{(?)3}{2}$ | |
| D | F | | $\frac{9}{2} \ \frac{17}{2} \ \frac{12}{2}$ | |
| $4\frac{1}{2}$ | $5\frac{1}{2}$ | $\frac{(\overline{0} \ 1 \ 2 \ 3 \ 4) \ \overline{10} \ 11 \ 12 \ 13 \ 14 \ 15 \ 16 \ 17 \ 18}{10}$ | $\frac{(0)6}{5?}$ | $\frac{(0)5}{4}$ |
| $4\frac{1}{2}$ | $4\frac{1}{2}$ | $\frac{(3 \ 6 \ 9 \ \overline{12}) \ 18 \ 21 \ 24 \ \overline{27} \ \overline{30} \ 33 \ 36 \ 39}{20}$ | | $\frac{(1)3?}{2}$ |
| $4\frac{1}{2}$ | $3\frac{1}{2}$ | $\frac{(\overline{0} \ 1 \ 2 \ 3) \ \overline{3} \ 4 \ 5 \ 6 \ 7 \ 8 \ \overline{9}}{4}$ | | $(0)2$ |
| $3\frac{1}{2}$ | $3\frac{1}{2}$ | $\frac{(1 \ 2 \ \overline{3}) \ \overline{3} \ 4 \ 5 \ \overline{6} \ 7 \ 8}{4}$ | | $\frac{(2)4}{3}$ |
| $2\frac{1}{2}$ | $2\frac{1}{2}$ | $\frac{(1 \ 2) \ \overline{1} \ 2 \ 3 \ 4}{2}$ | $\frac{(12)1234}{2}$ | $\frac{(12)1234}{2}$ |
| $1\frac{1}{2}$ | $2\frac{1}{2}$ | $\frac{(0 \ 1) \ \overline{1} \ 2 \ 3 \ 4}{2}$ | $\frac{(01)12}{2}$ | $\frac{(01)123}{2}$ |
| $2\frac{1}{2}$ | $1\frac{1}{2}$ | $\frac{(3) \ \overline{0} \ 3 \ 6}{2}$ | $\frac{(03)036}{2}$ | $\frac{(3)03}{2}$ |
| $1\frac{1}{2}$ | $1\frac{1}{2}$ | $\frac{(3) \ \overline{0} \ 3}{2}$ | $\frac{(3)3}{2}$ | $\frac{(3)03}{2} \ \frac{(3)03}{2}$ |
| $\frac{1}{2}$ | $1\frac{1}{2}$ | 0 | 0 | 0 |

¹⁾ The maxima of intensity are overlined.

TABLE II (Continued).

| Combination | | ZEEMAN effect | | | |
|----------------|----------------|---|--|--------------------------------|--|
| J | J | Calculated | | Observed (multiplet number) | |
| F | F | | | 10 | 11 |
| $5\frac{1}{2}$ | $5\frac{1}{2}$ | $\frac{(0) 7}{5}$ | | $\frac{(0) 3}{2}$ | $\frac{(0) 7}{5}$ |
| $5\frac{1}{2}$ | $4\frac{1}{2}$ | $\frac{(\bar{0} 1 2 3 4) 24 25 26 27 28 29 30 31 \bar{32}}{20}$ | | $\frac{(0) 3}{2}$ | $\frac{(0) 3}{2} \frac{(?) 7}{4} \frac{(0) 8}{5}$ |
| $4\frac{1}{2}$ | $4\frac{1}{2}$ | $\frac{(0) 27}{20}$ | | $\frac{(0) 3}{2}$ | $\frac{(?) 4}{3}$ |
| $4\frac{1}{2}$ | $3\frac{1}{2}$ | $\frac{(\bar{0} 2 4 6) 21 23 25 27 29 31 \bar{33}}{20}$ | | $\frac{(0) 3}{2}$ | $\frac{(0) 3}{2} \frac{(0) 3}{2} \frac{(?) 03}{2}$ |
| $3\frac{1}{2}$ | $3\frac{1}{2}$ | $\frac{(0) 5}{4}$ | | $\frac{(0) 5}{4}$ | $\frac{(?) 6}{4}$ |
| $3\frac{1}{2}$ | $2\frac{1}{2}$ | $\frac{(\bar{0} 1 2) 3 4 5 6 \bar{7}}{4}$ | | $\frac{(?) 3}{2}$ | $\frac{(?) 8}{5} \frac{(0) 4}{3}$ |
| $2\frac{1}{2}$ | $2\frac{1}{2}$ | $\frac{(0) 1}{1}$ | | $\frac{(0) 1}{1}$ | $\frac{(?) 1}{1}$ |
| $2\frac{1}{2}$ | $1\frac{1}{2}$ | $\frac{(0 1) 0 1 2}{1}$ | | $\frac{(0 1) 0 1 2}{1}$ | |
| $1\frac{1}{2}$ | $1\frac{1}{2}$ | 0 | | 0 | 0 |

TABLE III

| Observed combination multiplets ¹⁾ | | | |
|---|--------------|------------------|-------------|
| 1 $D'-d'$ | 7 $F'-d'$ | | |
| 2 $D'-D''$ | 8 $F'-D''$ | | |
| 3 $D'-D'''$ | 9 $F'-D'''$ | 16 $D'''-D^{IV}$ | |
| 4 $D'-F''$ | 10 $F'-F''$ | | 24 $f'-F''$ |
| 5 $D'-F'''$ | 11 $F'-F'''$ | 17 $F'''-D^{IV}$ | |
| 6 $D'-x$ | 12 $F'-x$ | | |
| | 13 $F'-f''$ | | 18 $f'-f''$ |
| 22 $D'-g'$ | 14 $F'-g'$ | | 19 $f'-g'$ |
| 21 $D'-y$ | 15 $F'-y$ | | 20 $f'-y$ |
| 24 $D'-d''$ (P' ?) | | | |

¹⁾ The combinations $D'-f''$, $f'-d'$ and $f'-D''$ seem to be also present in the iron-spectrum.

TABLE IV.

| 1 | D'_1 4½ 415.9 | D'_2 3½ 288.1 | D'_3 2½ 184.1 | D'_4 1½ 89.9 | D'_5 ½ |
|----------------------|--------------------|--------------------|--------------------|-------------------|-------------|
| d'_1 3½ 390.6 | 36766.84 | 36351.00 | 36062.94 | | |
| d'_2 2½ 252.0 | | 36741.55 | 36453.53 | 36269.68 | |
| d'_3 1½ | | | 36705.50 | 36521.39 | 36431.44 |
| 2 | D'_1 4½ | D'_2 3½ | D'_3 2½ | D'_4 1½ | D'_5 ½ |
| D''_1 4½ 344.0 | 39625.63 | 39209.92 | | | |
| D''_2 3½ 261.5 | 39969.72 | 39553.82 | 39265.71 | | |
| D''_3 2½ 173.2 | | 39815.25 | 39527.20 | 39343.20 | |
| D''_4 1½ 86.6 | | | 39700.36 | 39516.46 | 39426.34 |
| D''_5 ½ | | | | 39602.98 | |
| 3 | D'_1 4½ | D'_2 3½ | D'_3 2½ | D'_4 1½ | D'_5 ½ |
| D'''_1 4½ 240.2 | 25900.00 | 25484.03 | | | |
| D'''_2 3½ 199.5 | 26140.19 | 25724.24 | 25436.14 | | |
| D'''_3 2½ 139.7 | | 25923.77 | 25635.67 | 25451.45 | |
| D'''_4 1½ 71.1 | | | 25775.35 | 25591.23 | 25501.35 |
| D'''_5 ½ | | | | 25162.35 | |
| 4 | D'_1 4½ | D'_2 3½ | D'_3 2½ | D'_4 1½ | D'_5 ½ |
| F''_1 5½ 344.1 | 33695.37 | | | | |
| F''_2 4½ 289.2 | 34039.53 | 33624.72 | | | |
| F''_3 3½ 218.4 | 34328.72 | 33912.84 | 33623.60 | | |
| F''_4 2½ 144.9 | | 34131.31 | 33843.18 | 33659.02 | |
| F''_5 1½ | | | 33983.12 | 33803.97 | 33714.06 |

TABLE IV (Continued).

| 5 | D'_1 $4\frac{1}{2}$ | D'_2 $3\frac{1}{2}$ | D'_3 $2\frac{1}{2}$ | D'_4 $1\frac{1}{2}$ | D'_5 $\frac{1}{2}$ |
|--------------------------------|------------------------------|------------------------------|------------------------------|------------------------------|-------------------------|
| F'''_1 5 $\frac{1}{2}$ 292.3 | 26874.53 | | | | |
| F'''_2 4 $\frac{1}{2}$ 227.9 | 27166.82 | 26750.88 | | | |
| F'''_3 3 $\frac{1}{2}$ 164.9 | 27394.67 | 26978.76 | 26690.69 | | |
| F'''_4 2 $\frac{1}{2}$ 106.8 | | 27143.66 | 26855.57 | 26671.45 | |
| F'''_5 1 $\frac{1}{2}$ | | | 26962.43 | 26778.22 | 26688.31 |
| 6 | D'_1 $4\frac{1}{2}$ | D'_2 $3\frac{1}{2}$ | D'_3 $2\frac{1}{2}$ | D'_4 $1\frac{1}{2}$ | D'_5 $\frac{1}{2}$ |
| x_1 4 $\frac{1}{2}$ 411.2 | 33095.93 | 32679.98 | | | |
| x_2 3 $\frac{1}{2}$ 294.4 | 33507.13 | 33091.17 | 32803.10 | | |
| x_3 2 $\frac{1}{2}$ 145.4 | | 33385.54 | 32097.53 | 32913.43 | |
| x_4 2 $\frac{1}{2}$ 70.2 | | 33530.98 | 33242.94 | 33058.78 | |
| x_5 1 $\frac{1}{2}$ | | | 33313.08 | 33128.96 | 33039.01 |
| 7 | F'_1 5 $\frac{1}{2}$ 488.5 | F'_2 4 $\frac{1}{2}$ 351.3 | F'_3 3 $\frac{1}{2}$ 257.8 | F'_4 2 $\frac{1}{2}$ 168.9 | F'_5 1 $\frac{1}{2}$ |
| d'_1 3 $\frac{1}{2}$ 390.6 | | 29390.18 | 29038.87 | 28781.22 | |
| d'_2 2 $\frac{1}{2}$ 252.0 | | | 29429.48 | 29171.76 | 29002.73 |
| d'_3 1 $\frac{1}{2}$ | | | | 29423.76 | 29254.81 |
| 8 | F'_1 5 $\frac{1}{2}$ | F'_2 4 $\frac{1}{2}$ | F'_3 3 $\frac{1}{2}$ | F'_4 2 $\frac{1}{2}$ | F'_5 1 $\frac{1}{2}$ |
| D''_1 4 $\frac{1}{2}$ 344.0 | 32697.50 | 32249.05 | 31897.78 | | |
| D''_2 3 $\frac{1}{2}$ 261.5 | | 32593.04 | 32241.78 | 31983.99 | |
| D''_3 2 $\frac{1}{2}$ 173.2 | | | 32503.25 | 32245.56 | 32076.65 |
| D''_4 1 $\frac{1}{2}$ 86.6 | | | | 32418.71 | 32249.70 |
| D''_5 | | | | | 32336.56 |

TABLE IV (Continued).

| 9 | F'_1 5½ | F'_2 4½ | F'_3 3½ | F'_4 2½ | F'_5 1½ |
|----------------------|--------------|--------------|--------------|--------------|--------------|
| D'''_1 4½ 240.2 | 18981.72 | 18523.21 | 18171.92 | | |
| D'''_2 3½ 199.5 | | 18763.41 | 18412.11 | 18154.40 | |
| D'''_3 2½ 139.7 | | | 18611.62 | 18353.90 | 18184.98 |
| D'''_4 1½ 71.1 | | | | 18492.59 | 18353.66 |
| D'''_5 ½ | | | | | 18395.76 |
| 10 | F'_1 5½ | F'_2 4½ | F'_3 3½ | F'_4 2½ | F'_5 1½ |
| F''_1 5½ 344.1 | 26767.12 | 26318.65 | | | |
| F''_2 4½ 289.2 | 27111.27 | 26662.76 | 26311.47 | 26053.71 | |
| F''_3 3½ 218.4 | | 26952.00 | 26600.71 | 26342.98 | |
| F''_4 2½ 144.9 | | | 26819.15 | 26561.42 | 26392.53 |
| F''_5 1½ | | | | 26705.35 | 26537.43 |
| 11 | F'_1 5½ | F'_2 4½ | F'_3 3½ | F'_4 2½ | F'_5 1½ |
| F'''_1 5½ 292.3 | 19946.28 | 19497.77 | | | |
| F'''_2 4½ 227.9 | 20238.55 | 19790.04 | 19438.75 | | |
| F'''_3 3½ 164.9 | | 20017.93 | 19657.61 | 19408.90 | |
| F'''_4 2½ 106.8 | | | 19831.51 | 19573.77 | 19403.86 |
| F'''_5 1½ | | | | 19680.55 | 19511.63 |
| 12 | F'_1 5½ | F'_2 4½ | F'_3 3½ | F'_4 2½ | F'_5 1½ |
| x_1 4½ 411.2 | 26167.66 | 25719.16 | 25367.84 | | |
| x_2 3½ 294.4 | | 26130.35 | 25779.05 | 25521.31 | |
| x_3 2½ 145.4 | | | 26073.50 | 25815.77 | 25646.86 |
| x_4 1½ 70.2 | | | 26218.89 | 25961.16 | 25792.20 |
| x | | | | 26031.30 | 25863.38 |

TABLE IV (Continued).

| 13 | F'_1 5½ | F'_2 4½ | F'_3 3½ | F'_4 2½ | F'_5 1½ |
|------------------|----------------|----------------|----------------|----------------|---------------|
| f''_1 476.5 4½ | 29757.84 | 29309.36 | 28958.11 | | |
| f''_2 358.5 3½ | | 29785.88 | 29434.64 | 29176.87 | |
| f''_3 2½ | | | 20793.13 | 29535.23 | 29366.20 |
| 14 | F'_1 5½ | F'_2 4½ | F'_3 3½ | F'_4 2½ | F'_5 1½ |
| g'_1 388.4 5½ | 28450.93 | 28002.43 | | | |
| g'_2 311.8 4½ | 28839.29 | 28390.79 | 28039.50 | | |
| g'_3 3½ | | 27702.58 | 28351.32 | 28003.55 | |
| 15 | F'_1 5½ | F'_2 4½ | F'_3 3½ | F'_4 2½ | F'_5 1½ |
| y_1 474.9 5½ | 27854.20 | 27405.65 | | | |
| y_2 354.3 4½ | (?) | 27880.57 | 29529.29 | | |
| y_3 244.8 3½ | | 28234.85 | 27883.56 | 27625.84 | |
| y_4 2½ | | | 28128.34 | 27879.61 | 27701.70 |
| 16 | D'''_1 4½ | D'''_2 3½ | D'''_3 2½ | D'''_4 1½ | D'''_5 ½ |
| DIV_1 384.3 4½ | 18776.96 | 18536.78 | | | |
| DIV_2 272.6 3½ | 19161.28 | 18921.12 | 18721.57 | | |
| DIV_3 175.2 2½ | | 19193.65 | 18994.15 | 18854.45 | |
| DIV_4 86.0 1½ | | | 19169.41 | 19029.71 | 18958.64 |
| DIV_5 ½ | | | | 19115.72 | |

TABLE IV (Continued).

| 17 | F_1''' 5½ | 292.3 | F_2''' 4½ | 227.9 | F_3''' 3½ | 164.9 | F_4''' 2½ | 106.8 | F_5''' 1½ |
|------------------------|----------------|-------|----------------|-------|----------------|-------|----------------|-------|----------------|
| D_1^{IV} 4½ 384.3 | 17862.37 | | 17510.15 | | 17282.23 | | | | |
| D_2^{IV} 3½ 272.6 | | | 17894.46 | | 17666.58 | | 17501.71 | | |
| D_3^{IV} 2½ 175.2 | | | | | 17939.15 | | 17774.23 | | 17667.50 |
| D_4^{IV} 1½ 86.0 | | | | | | | 17949.53 | | 17842.76 |
| D_5^{IV} ½ | | | | | | | | | 17928.69 |
| 18 | f_1' 4½ | 584.7 | f_2' 3½ | 407.6 | f_3' 2½ | | | | |
| f_1'' 4½ 476.5 | 24709.51 | | 24125.21 | | | | | | |
| f_2'' 3½ 358.5 | 25186.48 | | 24601.78 | | 24194.19 | | | | |
| f_3'' 2½ | | | 24950.20 | | 24552.57 | | | | |
| 19 | f_1' 4½ | | f_2' 3½ | | f_3' 2½ | | | | |
| g_1' 5½ 388.4 | 23402.96 | | | | | | | | |
| g_2' 4½ 311.8 | 23791.33 | | 23206.58 | | | | | | |
| g_3' 3½ | 24103.11 | | 23518.41 | | 23110.76 | | | | |
| 20 | f_1' 4½ | | f_2' 3½ | | f_3' 2½ | | | | |
| y_1 5½ 474.9 | 22806.18 | | | | | | | | |
| y_2 4½ 354.3 | 23281.05 | | 22696.40 | | | | | | |
| y_3 3½ 244.8 | 23635.39 | | 23050.64 | | 22643.06 | | | | |
| y_4 2½ | | | 23295.48 | | 22887.83 | | | | |

TABLE IV (Continued).

| | | | | | | | | | |
|-----------------------|--------------|--------------|--------------|--------------|--------------|-------|--------------|------|-------------|
| 21 | D'_1 4½ | 415.9 | D'_2 3½ | 288.1 | D'_3 2½ | 184.1 | D'_4 1½ | 89.9 | D'_5 ½ |
| y_1 5½ 474.9 | 34782.2 | | | | | | | | |
| y_2 4½ 354.3 | 35257.1 | 34841.2 | | | | | | | |
| y_3 3½ 244.8 | 35611.4 | 35195.5 | 34907.5 | | | | | | |
| y_4 2½ | | 35440.3 | 35152.3 | 34968.2 | | | | | |
| 22 | D'_1 4½ | D'_2 3½ | D'_3 2½ | D'_4 1½ | D'_5 ½ | | | | |
| g'_1 5½ 388.4 | 35379.0 | | | | | | | | |
| g'_2 4½ 311.8 | 35767.4 | 35351.6 | | | | | | | |
| g'_3 3½ | 36079.3 | 35663.4 | 35375.4 | | | | | | |
| 23 | f'_1 4½ | f'_2 3½ | f'_3 2½ | | | | | | |
| F''_1 5½ 344.1 | 21719.2 | | | | | | | | |
| F''_2 4½ 289.2 | 22063.3 | 21478.6 | | | | | | | |
| F''_3 3½ 218.4 | 22352.5 | 21767.8 | 21360.2 | | | | | | |
| F''_4 2½ 144.9 | | 21986.2 | 21578.6 | | | | | | |
| F''_5 1½ | | | 21723.5 | | | | | | |
| 24 | D'_1 4½ | D'_2 3½ | D'_3 2½ | D'_4 1½ | D'_5 ½ | | | | |
| d''_1 ? 3½ 412.9 | 29057.2 | 28641.56 | 28353.35 | | | | | | |
| d''_2 ? 2½ 263.8 | | 29054.22 | 28766.22 | 28582.12 | | | | | |
| d''_3 ? 1½ | | | 29029.75 | 28845.80 | 28755.79 | | | | |

Mathematics. — "*Representation of the Polar Triangles of a Conic Section on the Points of Space*". By Dr. G. SCHAAKE. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of May 31, 1924).

§ 1. Let a one-one correspondence be given between the points P of a conic k^2 lying in the plane α , and the points Q of a rational biquadratic twisted curve k^4). To a chord k of k^4 joining the two points Q_1 and Q_2 of this curve, we associate the point K of α where the tangents to k^2 at the points P_1 and P_2 corresponding resp. to Q_1 and Q_2 , cut each other.

We shall show that the correspondence (K, k) leads us to a representation of the polar triangles of a conic different from k^2 on the points of space and we shall investigate the latter representation.

§ 2. If we choose K on k^2 , P_1 and P_2 coincide in K and Q_1 and Q_2 in the point Q associated on k^4 to this point of k^2 . The conic k^2 is accordingly the locus of the images K of the tangents to k^4 . A tangent to k^2 is the image of the cubic cone which projects k^4 out of the point Q associated to the point of contact P .

The quadruples of points of k^2 that correspond to the quadruples of points of k^4 in which this curve is intersected by planes, form an involution I_4^4 of the fourth order and the third rank. The triples of points which together with a given point of k^2 form quadruples of I_4^4 , define a cubic involution I_3^3 of the second rank. The neutral pair of points of I_3^3 forms with P a neutral triple of points of I_4^4 . Hence the neutral triples of I_4^4 define a cubic involution of the first rank, I^3 . I^3 consists of the triples of k^2 that are associated to the triples of k^4 lying on trisecants of the latter curve. A trisecant of k^4 , which may be considered as a triple chord, is represented on the three angular points of the triangle circumscribed to k^2 the sides of which touch k^2 in the points of a triple of I^3 . As I^3 has two pairs in common with the quadratic involution of the pairs of

¹⁾ The properties of this curve which we shall mention in this paper, and the corresponding proofs are to be found in the thesis for the doctorate of Dr. D. J. E. SCHREK, "Rationale ruimtekrommen van den vierden graad", Utrecht, 1915.

points of k^2 in which lines cutting each other on a given line l of α , touch k^2 , the images of the trisecants of k^4 form a conic t^2 which is circumscribed to an infinite number of tangent-triangles of k^2 . The conics k^2 and t^2 cut each other in the four double points S_i of I^2 , which are the images K of the trisecants t_i of k^4 touching this curve. The other two image points of a trisecant t_i coincide in the point T_i where the tangent to k^2 at S_i cuts the conic t^2 besides. The tangent to t^2 at T_i touches k^2 at the point that is associated to the point of k^4 where t_i cuts this curve.

The pairs of points of k^2 which together with two points coinciding in a given point of this conic, form a quadruple of I_3^4 , define an involution with two double points. Accordingly the quadruples of I_3^4 the points of which coincide in pairs, form an involutorial (2,2)-correspondence $I(2,2)$. The pairs of $I(2,2)$ are associated to the pairs of points of k^4 where the double tangent planes σ of this curve touch k^4 . The planes σ envelop a developable surface Δ of the fourth class and of the sixth order. For the projection of k^4 out of an arbitrary point has four double tangents and the involutorial (2,2)-correspondence of the pairs of points where generatrices of Δ cut k^4 , has six pairs of points in common with the likewise involutorial (3,3)-correspondence of the pairs of points of intersection of k^4 with chords cutting a given straight line. Δ contains k^4 as a nodal curve. The generatrices of Δ are represented on the points K of a conic d^2 , which, however, in this case is not circumscribed to a tangent triangle of k^2 . The conics d^2 and k^2 cut each other in the four double points D_i of $I(2,2)$, i.e. the four quadruple points of I_3^4 to which the points of k^4 with stationary planes of osculation are associated. A trisecant of k^4 touching k^4 , is at the same time a generatrix of Δ as the plane through such a trisecant t_i containing also the tangent to k^4 at the point of intersection of t_i with k^4 , touches this curve at the point of contact as well as at the point of intersection of t_i . Hence d^2 and t^2 intersect each other in the four points T_i .

§ 3. If K moves on a straight line r of α , k describes a scroll the generatrices of which cut k^4 in the pairs of the involution I^2 which corresponds to the involution of the pairs of points of k^2 where two straight lines intersecting each other on r , touch k^2 . I^2 has three pairs of points in common with the involutorial (3,3)-correspondence in which the chords intersecting a given straight line, cut k^4 . Hence to a straight line r of points K there corresponds a cubic scroll q^3 of chords k , which has k^4 as a single curve.

The involution I^2 is defined by two pairs of points. Let k_1 and k_2 be the carriers of two pairs of points of I^2 . The chords of k^4 cutting k_1 , form a cubic surface with k^4 as a single curve as from the surface of the ninth order of the chords of k^4 cutting an arbitrary straight line l , which has k^4 as a triple curve, two cubic cones through k^4 split off. Outside k^4 the chord k_2 cuts one of the generatrices of this cubic surface, so that there is one chord k_p of k^4 intersecting k_1 and k_2 outside this curve. The chords of k^4 intersecting k_p define an involution on k^4 of which the pairs of points are the intersections of planes through k_p with k^4 and which has two pairs of points in common with I^2 and is therefore identical with I^2 . *The scroll q^3 associated to a straight line r of α , is accordingly the scroll of the chords of k^4 that cut a given chord k_p . Inversely the chords of k^4 intersecting a given chord k_p , are represented on the points K of a straight line. This line is the locus of the points of intersection of pairs of tangents touching k^2 in pairs of points of the involution corresponding to the I^2 defined by k_p .*

To the points of intersection of r with k^2 , there correspond the generatrices of q^3 which touch k^4 and lie therefore in the two planes through k_p touching k^4 outside k_p . The pole K of r relative to k^2 is therefore the image of the line k joining the points of contact of the tangents to k^4 that cut the said k_p . The points of intersection with t^2 are images of the trisecants of k^4 through the points of intersection of k_p with this curve, and the points of intersection with d^2 are the images K of the generatrices of Δ which cut k_p outside k^4 .

The chord k_p is a nodal line of q^3 , as two more chords of k^4 pass through a point of k_p . If we associate to each other the image points K of two chords of k^4 passing through the same point of k_p , we get an involution on r . As this involution has two double points, k_p contains two points for which the chords of k^4 through them coincide. For each of the said two points the plane through the two chords passing through a point of k_p , touches k^4 in the points of intersection of k^4 with the straight line in which the two chords through such a point coincide, so that this straight line is a generatrix of Δ . The double points of the involution defined on r , are accordingly the intersections of r with d^2 . The images K of two chords of k^4 cutting each other outside k^4 , lie on the line r corresponding to the third chord through their point of intersection, considered as a chord k_p , and are associated to each other relative to d^2 .

Two chords k of k^4 intersecting each other outside k^4 , are accordingly represented on two points K of α , which are associated to each other relative to d^2 .

Hence:

The three chords of k^4 passing through an arbitrary point of space outside k^4 , are always represented on the angular points of a polar triangle of d^2 .

It is easily seen that also the inverse of this theorem holds good.

The angular points of the common polar triangle of d^2 and k^2 are the images of the cardinal chords or double osculation-chords of k^4 lying in the planes of osculation of the points of k^4 where each of the said chords rests on k^4 . These chords pass through one point, the so-called cardinal point of the curve.

§ 4. We arrive at the representation announced in § 1 by associating to each polar triangle φ of d^2 the point F of space where the chords k corresponding to the angular points K_i of φ , cut each other.

We remark that d^2 can coincide with an arbitrary conic of α . For we can also arrive at the correspondence (K, k) by starting from a one-one correspondence between the points of d^2 and the generatrices of the rational scroll Δ and by associating to a chord k of k^4 the point of intersection K of the tangents to d^2 at the points corresponding to the generatrices of Δ which cut k outside k^4 .

To a curve of points F there corresponds a system of ∞^1 polar triangles φ . The order of such a system is the number of its polar triangles which have an angular point on a given line of α . The order of a system of ∞^2 polar triangles, which is therefore represented on a surface of points F , is the number of its individuals which have an angular point given in α .

§ 5. For the representation (φ, F) the points of k^4 are singular. If a point F' approaches a point F of k^4 , the three chords of k^4 through that point get into the trisecant of k^4 through F and the two chords of k^4 through F lying in a tangent plane to k^4 at F . For a chord through F' for which F' and a point of intersection with k^4 approach each other, must finally coincide with one of the two chords through F which lie in the final position of the plane through F' and the tangent at F and the third chord through F' becomes the trisecant through F . Accordingly there correspond to F ∞^1 polar triangles of d^2 with one common angular point K_1 lying on t^2 whereas the other angular points of each are a pair of the involution defined by d^2 on the polar line r_1 of K relative to d^2 . This may also be seen in the following way. The line r_1 is a tangent of k^2 and its points K are therefore the images of the chords of k^4 passing through

the point Q of this curve which corresponds to the point of contact P of r_1 . Each pair of the involution defined on r_1 by d^2 , corresponds to two chords lying in tangent planes to k^4 at Q , because the involution arising on r_1 if we associate the image points of each of such a pair of chords to each other, must also have its double points in the points of intersection of r_1 with d^2 .

To a point F of k^4 there corresponds a system of ∞^1 polar triangles of the first order.

The ∞^2 polar triangles associated to points F of k^4 , are those for which one angular point lies on t^2 and the subtending side touches k^2 . They form a system Σ of the second order.

There exists therefore a rational quartic of singular points F . To each of these points there corresponds a linear system of ∞^1 polar triangles. The polar triangles associated to the points of k^4 , form a quadratic system of ∞^2 individuals.

The polar triangles of d^2 circumscribed to k^2 and therefore inscribed in t^2 , are singular for our representation. For if we choose for φ one of these triangles, the three chords k_i corresponding to the angular points K_i , coincide in a trisecant of k^4 , on which the point F may be chosen at random.

There exists accordingly a quadratic system S of ∞^1 singular polar triangles to each of which there corresponds a straight line of points F . The locus of the points F which are associated to all the individuals of S , is the quadratic surface τ^2 of the trisecants of k^4 .

§ 6. The polar triangles φ of a linear system of ∞^1 individuals have one common angular point and the other two lie on the polar line of this point relative to d^2 . The point of intersection of the three chords k_i which correspond to the angular points K_i of such a triangle, lies therefore always on the chord k_1 associated to the fixed angle K_1 .

A linear system of ∞^1 polar triangles is accordingly represented on a chord of k^4 .

The points of intersection of this chord with k^4 are the images of the triangles of the system which have an angular point in one of the points of intersection of the polar line of K_1 relative to d^2 , with t^2 .

The image points of the polar triangles of d^2 which have an angular point on a given straight line r of α , lie on the chords k of k^4 that are associated to the points K of this straight line. *To such a system of ∞^2 polar triangles there corresponds therefore a cubic scroll ϱ^3 passing through k^4 .*

To the points F of a straight line l of space there correspond ∞^1

polar triangles of d^2 , which form a system A of the third order, because l has three points of intersection with a surface ϱ^3 and A contains therefore three triangles with an angular point on a given straight line r . To the points of intersection of l and τ^2 there correspond two triangles inscribed in t^2 and circumscribed to k^2 . The locus of the angular points of the triangles of A is a cubic λ^3 circumscribed to an infinite number of polar triangles of d^2 , but also to an infinite number of complete quadrilaterals the sides of which touch k^2 . The six angular points of any of these quadrilaterals are the images K of the chords k of k^4 in a plane through l .

A straight line l of points F is the image of a cubic system A of ∞^1 polar triangles of d^2 which has two individuals in common with S .

If l cuts k^4 , a linear system of polar triangles splits off from A . There remains a quadratic system that has one individual in common with S . The locus of the angular points is in this case a conic circumscribed to an infinite number of polar triangles of d^2 but also to an infinite number of tangent-triangles of k^2 . The angular points of the former are points K of the chords through an arbitrary point of l , those of the latter are points K of the chords which lie in an arbitrary plane through l but do not intersect l on k^4 .

The polar triangles φ which are associated to the points F of a plane V , form a linear system Φ of ∞^2 individuals. For there is one polar triangle that has an angular point in a given point K of α , to wit the triangle which is represented on the point of intersection of the chord k corresponding to K with V . As each trisecant of k^4 has one point in common with V , Φ contains all the polar triangles of d^2 inscribed in t^2 .

Consequently a plane V is the image of a linear system of ∞^2 polar triangles that contains S .

§ 7. A twisted curve of the order n intersecting k^4 in m points, has in common with a surface ϱ^3 and with τ^2 resp. $3n-m$ and $2n-m$ points that are not singular for the representation.

Hence:

A curve of the order n cutting k^4 in m points, is the image of a system of ∞^1 polar triangles which has the order $3n-m$ and contains $2n-m$ individuals of S .

If inversely we have a system of ∞^1 polar triangles of the order p which contains q individuals of S , we find if we substitute for p and q resp. $3n-m$ and $2n-m$ and if we resolve n and m out of the equations arising in this way:

A system of ∞^1 polar triangles of the order p which has q individuals in common with S , is represented on a curve of the order $p-q$ cutting k^4 in $2p-3q$ points.

The image of a system of ∞^1 polar triangles of the order p which is general relative to S , is accordingly a curve of the order p which cuts k^4 in $2p$ points.

A surface of the order v containing k^4 as a μ -fold curve, has in common with a bisecant of k^4 $v-2\mu$ points and with a trisecant $v-3\mu$ points which are not singular for the representation.

A surface of the order v of which k^4 is a μ -fold curve, is therefore the image of a system of ∞^3 polar triangles of the order $v-2\mu$ which contains S ($v-3\mu$) times.

Inversely it is easily seen that:

A system of ∞^3 polar triangles of the order π containing S κ times, is represented on a surface of the order π of which k^4 is a $(\pi-\kappa)$ -fold curve.

Consequently a system of ∞^3 polar triangles of the order π which is general relative to S , is associated to a surface of the order 3π which contains k^4 π times.

§ 8. The images of a system of ∞^1 polar triangles of the order p that is general relative to S , and of a similar system of ∞^3 polar triangles of the order π , which are resp. a curve of the order p cutting k^4 in $2p$ points, and a surface of the order 3π with a π -fold curve in k^4 , have in common $3p\pi-2p\pi=p\pi$ points that are not singular for the representation.

Accordingly a system of ∞^1 polar triangles of the order p and a system of ∞^3 polar triangles of the order π relative to the same conic, have $p\pi$ polar triangles in common.

The images of two general systems of ∞^3 polar triangles of the order π and π' have in common besides k^4 a curve of the order $9\pi\pi'-4\pi\pi'=5\pi\pi'$ cutting k^4 in $10\pi\pi'$ points, because a point of intersection outside k^4 of this curve and τ^2 would cause these two surfaces to have a trisecant of k^4 in common so that the corresponding systems would contain the same polar triangle of S .

Hence:

Two systems of the order π and π' of ∞^3 polar triangles of the same conic have in common a system of ∞^1 individuals of the order $5\pi\pi'$.

The polar triangles of a system Π of ∞^3 individuals which have one angular point on a given straight line r and which therefore also belong to a linear system of ∞^2 polar triangles, form accordingly a system of ∞^2 individuals of the order 5π .

The locus of the angular points of the polar triangles of the latter system consists of the line r , counted π times, and a curve of the order 4π .

Consequently if a point K_1 describes a straight line r , the angular points different from K_1 of the polar triangles of π that have an angular point in K_1 , describe a curve of the order 4π .

From the former of the two theorems derived in this § there follows:

Three systems of ∞^2 polar triangles of the same conic which have resp. the order π , π' , and π'' , have $5\pi\pi'\pi''$ individuals in common.

§ 9. A system Π of ∞^2 polar triangles of the order π gives an involutorial $(2\pi, 2\pi)$ -transformation in α if we associate the angular points of the same triangle of Π to each other. This transformation contains two kinds of branch-points.

In the first place we have a single branch-point E_1 if the two angular points different from E_1 of a polar triangle of Π which has an angle in E_1 coincide. In this case the coinciding angular points lie on d^3 . Such polar triangles are represented on points for which two of the three chords of k^4 through them coincide; these points lie therefore on the developable surface Δ which according to § 2 is of the sixth order and has k^4 as a nodal curve.

The image of Π has in common with the surface Δ outside k^4 a curve of the order $18\pi - 8\pi = 10\pi$, which cuts k^4 in 20π points so that the polar triangles of Π with two coinciding angular points, form a system of the order 10π . Each point of k^3 contains two angular points of π of these triangles, so that k^3 splits off 2π times from the locus of the angular points of the triangles of the latter system. There remains accordingly a curve of the order 6π for the angular points that do not lie on k^3 .

The latter may also be seen in the following way. From the curve of intersection of the surface Δ with a surface ϱ^3 there split off k^4 , counted double, and the two straight lines of Δ that are torsal lines of ϱ^3 , also twice each, because along these lines Δ and ϱ^3 have the same tangent planes. There remains accordingly a curve of the sixth order which cuts k^4 twelve times and which is the image of the system of polar triangles of d^2 with two coinciding angles which have their third angular point on a given straight line r . This curve intersects the surface corresponding to Π in 6π points not singular for the representation, so that Π contains 6π single branch-points lying on a given straight line.

Consequently the locus of the single branch-points of the involutorial

transformation which is defined by a system of ∞^2 polar triangles, is a curve of the order 6π .

The polar triangles of which the three angular points coincide, which can only happen in a point of d^2 , are represented on points for which the three chords of k^4 passing through them, coincide. The projection of k^4 out of such a point on an arbitrary plane has a node of osculation in the point of intersection of the line in which the three chords coincide, with the plane of projection. In the tangent at this point there coincide three double tangents of the projection¹⁾. Hence the polar triangles with three coinciding angular points are represented on the points of the cuspidal curve σ of Δ . This curve passes through the points of the nodal curve k^4 of Δ that have two coinciding tangent planes to Δ . For such a point the involution in which planes through the tangent to k^4 cut k^4 , becomes parabolic. In this case all the pairs of points must have one point in common; hence the tangent to k^4 in a common point of σ and k^4 must cut the latter curve in one more point. As we saw in § 2 there are four points on k^4 that have this property. As the two generatrices of Δ through such a point coincide in the tangent to k^4 , σ and k^4 have there the same tangent. Therefore σ must have a cusp in such a point²⁾. A projection of σ is a rational curve of the sixth class with six double tangents and four inflectional tangents. According to the second formula of PLÜCKER the order of σ is therefore equal to:

$$6 \times 5 - 2 \times 6 - 3 \times 4 = 6.$$

For the determination of this order we have not made use of the number of cusps. We can also determine this number by the aid of the first or the third formula of PLÜCKER if we remark that the number of nodes of a projection of σ is ten. Both formulas give indeed that σ must have four cusps.

As σ has three coinciding points of intersection with any tangent plane to k^4 in each of the four points which this curve has in common with k^4 , σ and the surface corresponding to Π , must have 6π points of intersection that are not singular for the representation.

Accordingly a system of ∞^2 polar triangles of the order π has 6π triangles with three coinciding angular points.

For the linear system of the polar triangles that have an angular point on a given straight line, these six triangles coincide apparently in groups of three in the points of intersection of r with d^2 .

¹⁾ SALMON-FIEDLER, Höhere ebene Kurven, Leipzig 1882. p. 279, 3.

²⁾ ZEUTHEN, Abzählende Methoden, p. 143.

Hence the cubic scrolls through k^4 osculate the cuspidal curve of Δ twice outside k^4 .

§ 10. The involutorial transformation defined by Π , has a double branch-point in a point K of α , if two of the π polar triangles of Π that have an angular point in K , coincide. The chord k of k^4 corresponding to K , must touch in this case the image surface ω of Π outside k^4 . We ask how many of such chords there pass through a given point O of k^4 .

We associate to each other the points in which a chord through O cuts the surface ω outside k^4 . We get in this way ∞^1 pairs of points (P, Q) to which we apply the formula of SCHUBERT¹⁾

$$\varepsilon = p + q - g.$$

Here p and q are the numbers of pairs (P, Q) for which resp. P and Q lie in a given plane. The cubic cone projecting k^4 out of O , has besides k^4 an intersection with ω of the order 5π . To each of the 5π points in which a given plane cuts this curve, there correspond $\pi - 1$ points which together with such a point of intersection form a pair (P, Q) . Hence $p = q = 5\pi(\pi - 1)$. Further g is the number of pairs (P, Q) for which the carriers cut a given straight line g . Each of the three lines of the cubic cone projecting k^4 out of O , carries $\pi(\pi - 1)$ pairs of points. Hence $g = 3\pi(\pi - 1)$.

We find that the number ε of the coincidences (P, Q) is equal to $7\pi(\pi - 1)$. Some of these coincidences lie apparently in the points O_1 and O_2 which the trisecant through O has besides in common with k^4 . The intersection outside k^4 of the cubic cone and ω has apparently in each of these points a π -fold point. A plane infinitely near to O_1 contains accordingly π points P infinitely near to O_1 , to each of which there correspond $\pi - 1$ similar points Q . Consequently in each of the points O_1 and O_2 there lie $\pi(\pi - 1)$ coincidences (P, Q) . The carriers of the remaining $5\pi(\pi - 1)$ coincidences are chords of k^4 through O which touch ω outside k^4 .

The point K corresponding to one of these chords, is the angular point of one polar triangle belonging to the linear system that corresponds to O . K lies accordingly on the straight line of the free angular points of this system. Hence:

The locus of the double branch-points of the involutorial transformation that is defined by a system of ∞^2 polar triangles of the order π , is a curve of the order $5\pi(\pi - 1)$.

¹⁾ Kalkül der abzählenden Geometrie, p. 44.

§ 11. We assume again a system Π of ∞^1 polar triangles of the order π . If a point K_1 describes a straight line r , the angular points different from K_1 of the polar triangles of Π which have an angular point in K_1 , describe a curve $r^{4\pi}$, according to § 9. The pairs of points of this curve which correspond to the same point K_1 of r , lie on straight lines through the pole R of r . $r^{4\pi}$ has a 2π -fold point in R as the π triangles of Π that have an angular point in R , have 2π more angular points on r .¹⁾

$r^{4\pi}$ has further π -fold points in the points of intersection of r with d^2 , because the triangles corresponding to these angular points have a second angular point in such an intersection¹⁾).

Accordingly

$$4\pi(4\pi-1) - 2\pi(2\pi+1) - 2\pi(\pi-1) = 10\pi^2 - 10\pi$$

tangents may be drawn out of R to $r^{4\pi}$. The points of contact of 6π of these lines are coinciding angular points of polar triangles of Π and lie therefore on d^2 . The remaining $10\pi(\pi-1)$ coincide in pairs in $5\pi(\pi-1)$ double tangents and are sides of individuals of Π in which two triangles associated to the same point of r , coincide.

In this way we have found a check on the results of the two preceding §§ and we have found at the same time:

The curve corresponding in a system Π to a straight line r , is of the genus $5\pi^2 - 4\pi + 1$.

We remark that our results satisfy the formula of ZEUTHEN for the genus, applied to the correspondence between the points of r and of $r^{4\pi}$.

§ 12. The polar triangles that have an angular point on a curve σ^π of the order π given in α , form a system of ∞^1 individuals of the order π . For the polar line of an arbitrary point K_1 of α cuts σ^π in π points and each of these points forms with K_1 a pair of angular points of a polar triangle belonging to the said system. The image is a scroll consisting of the chords of k^4 which correspond to the points of σ^π . According to § 8 it is a surface of the order 3π that has a π -fold curve $\omega^{3\pi}$ in k^4 .

The systems corresponding in this way to two curves σ^π and $\sigma^{\pi'}$, have in common according to § 9 a system of ∞^1 individuals, which is of the order $5\pi\pi'$. The $\pi\pi'$ linear systems of the polar triangles that have an angular point in one of the points of intersection of the given curves, split off from this system. The locus of the angular

¹⁾ This may also be seen from the representation.

points of the remaining system of the order $4\pi\pi'$ contains o^π and $o^{\pi'}$, resp. π' and π times. There remains therefore a curve of the order $2\pi\pi'$.

For the polar triangles that have an angular point on each of two given curves of the order π and π' , the locus of the third angular points is a curve of the order $2\pi\pi'$.

Hence:

There are $2\pi\pi'\pi''$ polar triangles that have an angular point on each of three given curves of the order π , π' and π'' .

§ 13. If we take for o^π a conic o^2 , there corresponds to it a scroll ω^6 that has k^4 as a nodal curve. Let us now assume that o^2 is circumscribed to one polar triangle of d^2 . To the angular points of this polar triangle there correspond three chords k through the same point F_1 which is a triple point of ω^6 . A plane through an arbitrary generatrix of ω^6 has also in common with this surface a curve of the fifth order which cuts this generatrix besides in the point of contact of the plane and in the two points of intersection with k^4 , in two more points.

Two intersecting generatrices k of ω^6 correspond to two points K of o^2 associated to each other relative to d^2 , and to each point of o^2 there correspond indeed two points of the same conic associated to it relative to d^2 . Consequently besides k^4 ω^6 has another nodal curve. Let us assume a point F_2 on this curve. Among the ∞^6 cubic surfaces through k^4 there is one that has a conical point in F_1 and that contains the straight line $F_1 F_2$. The tangent plane at F_2 to this surface cuts it besides along a conic that passes through F_1 and F_2 and that cuts k^4 four times. We can at once point out thirteen points of intersection of this conic and ω^6 so that the conic lies entirely on ω .

Now the surface of the chords of k^4 that cut this conic outside k^4 , has the order six, as it intersects the surface of the ninth order of the chords of k^4 that meet a given straight line, in six points outside k^4 . Our surface ω^6 is accordingly identical with the surface of the chords of k^4 which cut a conic intersecting k^4 four times, and contains therefore a conic of triple points. To the three chords through a triple point of ω^6 there correspond three points of o^2 that are the angular points of a polar triangle of d^2 . *By means of our representation we have found a proof of the theorem that a conic that is circumscribed to one polar triangle of another one, has an infinite number of inscribed polar triangles of the latter conic.*

We see that the property according to which the angular points

of two polar triangles of the same conic lie on a conic, is equivalent to the one according to which there passes through two given points a conic that cuts k^4 four times. According to § 8 there corresponds to this conic a quadratic system that contains the two polar triangles which are represented on the given points.

§ 14. The triples of angular points of the individuals of a linear system Π of ∞^2 polar triangles of α , are triples of a cubic involution of the first rank in α . The triples of such an involution may be defined on α by the points of intersection with the twisted cubics of a congruence of REIJE which pass all through five given points. The involution has been investigated by Prof. VAN DER WOUDE¹⁾ independently from this definition.

According to § 8 the image of an arbitrary linear system of ∞^2 polar triangles is a cubic surface through k^4 . There are ∞^6 of such surfaces.

A linear system of ∞^2 polar triangles relative to a given conic is therefore defined by six of its triangles.

If the conic is not given we have:

In a given plane there lie ∞^{11} linear systems of ∞^2 polar triangles.

If Π contains the polar triangles of d^2 inscribed in t^2 , the system is represented on a plane. The properties of an arbitrary system Π may be derived from this representation. For we can always consider the cubic surface representing a general system Π , as belonging to a tissue of cubic surfaces that has as base curve k^4 and a conic p^2 which cuts k^4 four times. Now there exists a cubic transformation (F, F') which represents these surfaces on the planes of space. For k^4 and p^2 form together a possible degeneration of a twisted curve of the order six and the genus three which is the base curve of the tissue of the cubic surfaces on which the planes are represented through a cubic transformation²⁾. Among the points F' there are two curves of singular points. To the points of one of them there correspond the trisecants of k^4 , to the points of the other one the chords of k^4 that intersect p^2 . As an arbitrary cubic surface of the tissue contains two trisecants of k^4 and four chords of k^4 which cut p^2 , the singular curve corresponding to the surface of trisecants of k^4 is a conic p'^2 , the other one is a rational quartic k'^4 which has four points in common with the singular conic, to wit the points corresponding to the trisecants of k^4 through the points of

¹⁾ These Proceedings Vol. XII, p. 751.

²⁾ Cf. STURM: Geometrische Verwandtschaften, p. 370 and 392.

intersection of p^2 and k^4 . Together with the representation on a plane of a cubic surface ω^3 of the tissue in consideration, there is also given the representation on this plane of the system Π corresponding to ω^3 , which representation has only singular points in the points of intersection of the plane with the quartic of singular points F' , as to the points of intersection with the singular conic trisecants of k^4 , hence polar triangles of S , are associated. A closer examination shows that this representation is of the same nature as the representation on a plane of a system Π containing S , arising from the correspondence (φ, F') .

The representation (φ, F') differs from (φ, F) only in this respect that in the former the quadratic system of ω^1 polar triangles associated to p^2 , takes the place which S has in the latter.¹⁾

We find accordingly that t^2 may be chosen along an arbitrary conic circumscribed harmonically to d^2 (and consequently k^2 along an arbitrary conic inscribed harmonically in d^2).

§ 15. Let us now consider a system Π that is represented on a plane ω . The four points of intersection T_i of ω with k^4 and the six chords of k^4 in ω give ten linear systems of polar triangles belonging to Π , each of which is defined by a point S and a straight line s , so that S is an angular point of all the polar triangles of such a system and the other angular points lie on the straight line s . The four points S_i of the systems corresponding to the points of intersection of ω with k^4 , lie on t^2 . To these there correspond four lines s_i touching k^2 . A straight line $T_i T'_k$ of ω represents the linear system of Π that has a triangle in common with each of the systems corresponding to T_i and T'_k and of which the point S_{ik} lies accordingly in the intersection of s_i and s_k , the straight line s_{ik} along $S_i S_k$.

The line s_{14} of the system corresponding to $T_1 T_4$ passes through S_1 and S_4 but also through S_{23} , because $T_1 T_4$ and $T_2 T_3$ have a point of intersection that is not singular for the representation. In the same way it appears that $S_2 S_4$ joins the points S_2 and S_1 , and $S_2 S_3$ the points S_2 and S_{13} , and that these straight lines pass through S_4 . Accordingly the triangle $S_1 S_2 S_3$ and the trilateral $s_1 s_2 s_3$ associated to it by a polar correspondence, lie perspectively. The joins of corresponding angular points are lines s of the linear systems belonging

¹⁾ As one trisecant and two chords intersecting p^2 pass through a point of k^4 there corresponds to this point a chord of k^4 intersecting p^2 . To a point of p^2 through which there pass no trisecant and three bisecants, there corresponds a trisecant of k^4 .

to Π , the points of intersection of the corresponding sides, i.e. the poles of these lines s , are the corresponding points S . The ten points S are accordingly the angular points of the triangles $S_1 S_2 S_3$ and $s_1 s_2 s_3$, the points of intersection of the corresponding sides of these triangles, and the centre of perspectivity S_4 ; the ten straight lines s are the sides of the said triangles, the joins of corresponding angular points, and the axis of perspectivity s_4 .

Of the cubic involution defined by Π the points S are apparently singular points, the lines s singular lines. They form a *configuration of DESARGUES*.

As each point of the plane ω belongs to a conic of the pencil that has the points T_i as base points, Π consists of the individuals of the quadratic systems corresponding to the pencil in question. The loci of the angular points of these systems are the conics of the pencil that has S_1, S_2, S_3 and S_4 as base points, which form a polar quadrangle of d^3 . S_{12} is e.g. the pole of $S_1 S_2$ and S_{12} lies on the subtending side $S_3 S_4$ of the said quadrangle. *We see therefore that we can get any involution that may be derived from a linear system, by the aid of a pencil of conics that has the angular points of a polar quadrangle as base points, by associating to any a point of α the points where the polar line of the chosen point cuts the conic through it of the pencil in question.*

Inversely in this way we always get an involution of which the triangles of the point triples form a system Π because all the conics of a pencil that has the angular points of a polar quadrangle of d^3 as base points, are harmonically circumscribed to d^3 .

We find that a system Π may be produced in five such ways. For to a plane pencil of rays in the plane ω that has a point T_i for vertex, there corresponds also a pencil of quadratic systems of Π . The conics corresponding in this way to the lines of the plane pencil that has for instance T_1 for vertex, all pass through S_1, S_{23}, S_{34} and S_{14} , through the latter three points because a straight line of this plane pencil cuts $T_1 T_2, T_1 T_3$ and $T_1 T_4$ in points that are not singular for the representation¹⁾.

§ 16. If a point K_i describes a straight line r of α , the angular points K_2 and K_3 corresponding to K_1 in a linear system Π , describe according to § 12 a curve r^4 , of the order four and the genus two

¹⁾ A straight line of ω is the image of the system of inscribed polar triangles of a cubic that is circumscribed to the complete quadrilateral (s_1, s_2, s_3, s_4) . This gives a way of producing Π which has been indicated by Prof. JAN DE VRIES. These Proceedings XXI, p. 295. There are also five such ways.

that has a node in the pole R of r and that passes through the points of intersection of r with d^3 . This curve also passes through the points S of the configuration of DESARGUES corresponding to Π , because whenever K_1 lies in a point of intersection of a straight line s with r , one of the two points K_2 and K_3 gets into the corresponding point S . The tangents of r^4 at R are sides of the polar triangle of Π that has R as an angular point and they cut r in the other two angular points of this triangle, through which r also passes, because to one of these points, considered as a point K_1 , the other point corresponds as a point K_2 or K_3 . The curve r^4 has therefore the property that the nodal tangents at the node R cut it in two more points the join of which has two more points in common with r^4 , which separate the former two harmonically. It has been considered by Prof. v. D. WOUDE but not in this connection¹⁾.

The system of polar triangles of Π having an angular point on r , is represented on the intersection of the surface corresponding to Π and the surface φ^3 associated to r , that is a curve of the order five, k^5 , which has ten points in common with k^4 . We may consider the curve which is composed of k^4 and k^5 , as the base curve of a pencil of cubic surfaces to which there corresponds a pencil of systems Π that have in common the polar triangles with one angular point on r and two on r^4 . As in all these systems r is associated to the same curve r^4 , r^4 passes through the points S of all the configurations of DESARGUES that are defined by the singular elements of these systems.

To any point K of r^4 there corresponds a chord k of k^4 which cuts k^5 and does not lie on φ^3 . In the pencil in consideration there is one cubic surface which contains k . For the system Π of which this surface is the image, a point S lies in K . We see from this that any point of r^4 belongs to one of the configurations of DESARGUES inscribed in this curve.

Through a point of k^5 there passes one chord of k^4 belonging to φ^3 , and there are two chords through this point that do not lie on φ^3 . These chords are associated resp. to the angular point on r and to the angular points on r^4 of a polar triangle. As the chords through the same point of k^5 do not lie on the same cubic surface of the pencil, the angular point K_1 on r^4 of the polar triangle that has one singular point in a given point K_2 of r^4 and another one on r , is not a part of the configuration of DESARGUES to which K_2 belongs. Three

¹⁾ These Proceedings XXII, p. 645.

of the points of intersection of the polar line of K_2 with r^4 belong therefore to the same configuration of DESARGUES that is inscribed in r^4 as K_2 , the fourth is, the same as K_2 , an angular point of the polar triangle of which the third angle lies on r .

A similar curve may be produced by the aid of a pencil of conics that have the angular points of a polar quadrangle as base points. For it is locus of the points of intersection of the polar lines of the points of a straight line r with the conics of the pencil passing through these points. We may start from any pencil of conics as every quadrangle is a polar quadrangle relative to ∞^3 conics.

If we choose two arbitrary surfaces ω^3 , the intersection k^5 is the image of a system of ∞^1 polar triangles of the fifth order because the pencil defined by these surfaces, contains no scroll. This system consists of the polar triangles that are inscribed in a curve r^5 of the fifth order. This curve has no double points as k^5 and k^4 have no chord in common that cuts k^5 only outside k^4 , because this chord would lie on all surfaces of the pencil.

The curve r^5 contains the points S of all the individuals of the pencil of systems Π that is associated to the chosen pencil of surfaces ω^3 , as any chord of k^4 lying on one of the surfaces of the latter pencil, intersects k^5 . In the same way as in the preceding case we see that any point of r^5 is a part of a configuration of DESARGUES inscribed in this curve, and that three of the points of intersection of the polar line of a point K of r^5 with this curve belong to the same configuration of DESARGUES inscribed in r^5 , as K , and that the other two are, together with K , the angular points of a polar triangle inscribed in r^5 .

We have proved the existence of a curve r^5 of the order five and the genus six that has an infinite number of configurations of DESARGUES inscribed in it. The curve composed of r^4 and r is a special case of this curve.

Mathematics. — “*Ein topologischer Zerlegungssatz*”. By H. KNESER.
(Communicated by Prof. L. E. J. BROUWER).

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Die BROUWERSche Übertragung des JORDANSchen Kurvensatzes auf Räume beliebiger Dimensionenzahl besagt, dass der n -dimensionale Raum von jeder in ihm liegenden $(n-1)$ -dimensionalen geschlossenen Mannigfaltigkeit in zwei Teile zerlegt wird ¹⁾. Fragt man, welche geschlossenen n -dimensionalen Mannigfaltigkeiten diese Eigenschaft haben, so lautet die Antwort:

Satz 1. Eine geschlossene n -dimensionale Mannigfaltigkeit M^n ($n > 1$) wird dann und nur dann von jeder in ihr liegenden geschlossenen $(n-1)$ -dimensionalen Mannigfaltigkeit zerlegt, und zwar in zwei Teile, wenn ihre BETTische Zahl erster Dimension den Wert eins hat und die Torsionszahlen erster Dimension — wenn solche vorhanden sind — sämtlich ungerade sind.

Ich beweise diesen Satz im kombinatorischen Sinne, d. h. unter der Voraussetzung, dass M^n durch ein endliches Zellengebäude gegeben ist und M^{n-1} sich aus Zellen dieses Gebäudes zusammensetzt. In § 1 leite ich die Grundeigenschaften der “Umgebungs komplexe” ab, was bisher anscheinend noch nicht geschehen ist ²⁾; die folgenden Paragraphen erbringen schrittweise den Beweis für Satz 1.

Der Gedankengang des Beweises lässt sich etwa folgendermassen andeuten, wenn auch die Ausführung in Einzelheiten anders ausfällt. Ist eine M^n nicht zerlegende M^{n-1} gegeben, so verbinden wir zwei „nahe“ bei einem Punkte von M^{n-1} und „auf verschiedenen Seiten von M^{n-1} “ gelegene Punkte durch eine M^{n-1} nicht treffende Kurve und vervollständigen diese zu einer M^{n-1} in einem Punkte „durchschneidenden“ geschlossenen Kurve. Unter den Voraussetzungen von Satz 1 ist diese, eine geeignete ungerade Anzahl Male genommen, Rand einer in M^n gelegenen Fläche. Diese Fläche können

¹⁾ Beweis des JORDANSchen Satzes für den n -dimensionalen Raum, Math. Ann. 71 (1911), S. 314—319.

²⁾ Vgl. E. BILZ, Beiträge zu den Grundlagen der kombinatorischen Analysis situs (Math. Zschr. 18 (1923), S. 1—41).

wir so legen, dass sie M^{n-1} nur in geschlossenen und auf dem Rand der Fläche endenden ungeschlossenen Kurven trifft. Auf dem Rande liegt aber als einziger Treffpunkt der eine ungerade Anzahl Male zu zählende Durchstosspunkt, und jedes Mal geht von ihm eine ungerade Zahl Schnittkurven aus. Das geht nicht an, weil jede ungeschlossene Kurve zwei Endpunkte hat.

Sind die Voraussetzungen von Satz 1 nicht erfüllt, so hat die ABELSche Gruppe der Homologieklassen erster Dimension einen Charakter, der die Werte ± 1 und keine anderen annimmt. Es gelingt, einen M^n nicht zerlegenden Komplex $(n-1)^{\text{ter}}$ Dimension C^{n-1} zu konstruieren, derart dass ein geschlossener Weg, der C^{n-1} in k Punkten durchschneidet, als Element der Homologiengruppe der Charakter $(-1)^k$ hat. Vermöge dieser Eigenschaften gelingt es weiter, die Singularitäten des Komplexes C^{n-1} aufzulösen, ohne dass er dabei seine vorher genannten Eigenschaften verliert, d.h. aus ihm eine M^n nicht zerlegende M^{n-1} abzuleiten.

§ 1. Umgebungskomplexe.

Wir benutzen die Begriffe des n -dimensionalen Komplexes C^n , der n -dimensionalen Sphäre S^n , des n -dimensionalen Elementarraumes E^n und der internen Transformation (abgekürzt i. T.), die durch gleichzeitige vollständige Induktion nach n zu definieren sind ¹⁾.

Der Umgebungskomplex (abgekürzt UC) einer Zelle k -ter Dimension Z^k in einem Komplex C^n ($k < n$) entsteht, wenn wir jeder von Z^k berandeten ²⁾ Zelle Z^l ($k < l \leq n$) von C^n eine Zelle Z^{l-k-1} entsprechen lassen und die Berandungsbeziehungen in dem Sinne aufrecht erhalten, dass, wenn Z^l dem Rande von Z^m ($l < m$) angehört, auch die Z^l entsprechende Zelle Z^{l-k-1} dem Rande der Z^m entsprechenden Zelle Z^{m-k-1} angehören soll. Der UC von Z^k in C^n werde mit $UC_{Z^k}(C^n)$ bezeichnet. Um die Bezeichnung Umgebungskomplex zu rechtfertigen, müssen wir beweisen:

¹⁾ Man findet die Definitionen bei BILZ a.a.O., § 5. Wir weichen nur darin von den BILZ'schen Definitionen ab, dass wir die Forderungen 64b und c weglassen, die eine, weil sie in dem nötigen Umfange als Satz 2 bewiesen werden wird, die andere, weil auch ohne sie die wesentlichen Sätze bestehen bleiben.

²⁾ Der Kürze halber sagen wir, Z^k berandet Z^l , wenn Z^k der Rand- S^{l-1} von Z^l angehört, also nicht notwendig den vollen Rand von Z^l bildet.

Satz 2. $UC_{Z^k}(C^n)$ ist ein Komplex, d.h. der Rand jeder seiner Zellen Z^r ist eine S^{r-1} .

Fassen wir den UC unabhängig von seiner geometrischen Bedeutung nur als Zusammenstellung von Zellen auf, die durch Berandungsbeziehungen verknüpft sind, so gilt die Formel

$$UC_{Z^l}(C^n) = UC_{Z^{l-k-1}}(UC_{Z^k}(C^n)), \dots \quad (I)$$

wenn Z^k eine Randzelle von Z^l , also $k < l < n$, und Z^{l-k-1} die der Zelle Z^l auf $UC_{Z^k}(C^n)$ entsprechende Zelle ist. Um nämlich $UC_{Z^k}(C^n)$ zu bilden, behalten wir von C^n nur die von Z^k berandeten Zellen bei, vermindern die Dimension um $k+1$ und lassen die Berandungsbeziehungen ungeändert. Da Z^k als Randzelle von Z^l auch jede von Z^l berandete Zelle berandet, sind jedenfalls die von Z^l berandeten Zellen beibehalten worden, und zwar werden sie jetzt von Z^{l-k-1} berandet. Die rechte Seite der Gleichung (I) entsteht, wenn wir nur diese Zellen mit ihren Berandungsbeziehungen beibehalten und die Dimension je um $l-k-1+1$ vermindern. Im ganzen haben wir also genau die von Z^l berandeten Zellen beibehalten und ihre Dimension um $k+1+l-k-1+1=l+1$ vermindert. Das gibt aber genau $UC_{Z^l}(C^n)$.

Was ist nun der Rand von Z^{l-k-1} auf $UC_{Z^k}(C^n)$? Wir erhalten ihn, indem wir von den Zellen von $UC_{Z^k}(C^n)$ nur die Randzellen von Z^{l-k-1} , d.h. die aus den Randzellen von Z^l hervorgegangenen Zellen herausgreifen. Damit haben wir aber den UC von Z^k in der Z^l berandenden Sphäre S^{l-1} . Satz 2 ist also zurückgeführt auf

Satz 3(m).

$$UC_{Z^k}(S^m) = S^{m-k-1}.$$

Zugleich mit Satz 3(m) beweisen wir

Satz 4(m). Ist Z^k eine innere Zelle des Elementarraumes E^m , so gilt

$$UC_{Z^k}(E^m) = S^{m-k-1}.$$

Satz 5(m). Ist Z^k Randzelle von E^m , so gilt

$$UC_{Z^k}(E^m) = E^{m-k-1},$$

und zwar entspricht einer von Z^k berandeten inneren bzw. Randzelle von E^m eine innere bzw. Randzelle von E^{m-k-1} .

Man bestätigt die Sätze 3(m) bis 5(m) unmittelbar, wenn S^m bzw. E^m in der Normalgestalt vorliegt. Diese besteht bekanntlich ¹⁾ aus zwei Zellen jeder Dimension von der nullten bis zur m -ten, nur beim Elementarraum E^m nur einer Z^m , wobei jede Zelle jede von höherer Dimension berandet; und im Falle des Elementarraumes ist jede Zelle von niedrigerer als der m -ten Dimension Randzelle. Bildet man einen UC , so erhält man genau, wie es Satz 3(m) bzw. 5(m) behauptet, eine S^{m-k-1} bzw. einen E^{m-k-1} (Satz 4(m) kommt nicht zur Anwendung, da keine inneren Zellen vorhanden sind).

Es seien nun die Sätze 3(n), 4(n), 5(n) gültig für $n < m$ und für $n = m$ bei einer bestimmten Darstellung vom S^m bzw. E^m ; wir beweisen sie für jede S^m oder E^m , die aus der vorliegenden durch eine i.T. hervorgeht. Haben wir dies für $k = 0$ geleistet, so folgen die Sätze für grösseres k . Ist nämlich Z^0 ein Punkt auf dem Rande von Z^k so folgt aus (1), wenn Z^{k-1} die Z^k auf $UC_{Z^0}(S^m)$ bzw. $UC_{Z^0}(E^m)$ entsprechende Zelle ist,

$$UC_{Z^k}(S^m) = UC_{Z^{k-1}}(UC_{Z^0}(S^m))$$

bzw.:

$$UC_{Z^k}(E^m) = UC_{Z^{k-1}}(UC_{Z^0}(E^m)).$$

Sind die Sätze für Z^0 bewiesen, so steht rechts in der Klammer eine S^{m-1} oder ein E^{m-1} , je nachdem ob Z^0 innere oder Randzelle ist. Nach Satz 5(m), angewandt auf Z^0 , ist Z^{k-1} zugleich mit Z^k innere oder Randzelle. Nach Satz 3($m-1$), 4($m-1$) oder 5($m-1$) steht nun rechts eine S^{m-k-1} oder, wenn Z^k Randzelle von E^m ist, ein E^{m-k-1} , was zu beweisen war.

Der UC eines Punktes Z^0 ändert sich bei i.T. nur dann, wenn eine von Z^0 berandete Zelle Z^k transformiert wird. Es werde etwa Z^k in Z_1^k und Z_2^k geteilt durch eine Zelle Z^{k-1} , deren Rand eine auf der Randsphäre S^{k-1} von Z^k gelegene S^{k-2} ist, die S^{k-1} in E_1^{k-1} und E_2^{k-1} zerlegt, sodass Z_i^k ($i=1,2$) von Z^{k-1} und E_i^{k-1} begrenzt wird. UC_{Z^0} ändert sich nur, wenn Z^0 auf S^{k-2} liegt. Nach den Sätzen 3($k-1$), 3($k-2$), 5($k-1$) sind dann die Bilder von S^{k-1} , S^{k-2} , E_i^{k-1} auf UC_{Z^0} bezüglich Sphären bzw. Elementarräume \bar{S}^{k-2} , S^{k-3} , E_i^{k-2} . Auf UC_{Z^0} wird bei der i. T. eine von \bar{S}^{k-2} begrenzte Z^{k-1} ersetzt durch Z^{k-2} , begrenzt von S^{k-3} , und die Zellen Z_i^{k-1} ($i=1,2$), begrenzt von

¹⁾ Siehe BILZ, a. a. O., Nr. 79, 80.

Z^{k-2} und E_i^{k-2} , d. h. UC_{Z^0} erleidet eine i. T.: die Sätze 3 (n), 4 (n), 5 (n) bleiben gültig. Genau entsprechend beweist man, dass UC_{Z^0} eine i. T. erfährt, wenn bei der i. T. zwei Zellen vereinigt werden.

Wird bei der i. T. eine Strecke Z^1 in zwei Strecken Z_1^1 und Z_2^1 geteilt und ein Punkt Z^0 neu eingeführt, so erhalten wir UC_{Z^0} , indem wir in UC_{Z^1} die Dimension jeder Zelle um eins erhöhen und zwei Punkte, die Bilder von Z_1^1 und Z_2^1 , hinzufügen, die alle Zellen höherer Dimension beranden. Es ist also noch zu beweisen.

Satz 6 (n). Erhöht man in einer S^{m-1} bzw. einem E^{m-1} die Dimension jeder Zelle um eins, und fügt man zwei alle anderen Zellen berandende Punkte Z_1^0 und Z_2^0 hinzu, so ergibt sich eine S^m bzw. ein E^m .

Der Satz 6 (n) gilt offenbar für $n=1$ und wenn S^{n-1} bzw. E^{n-1} in der Normalgestalt vorliegt. Er sei bewiesen für $n < m$; dann zeigen wir, dass bei i. T. von S^{m-1} bzw. E^{m-1} auch S^m bzw. E^m eine i. T. erfährt. In der Tat, haben bei der Teilung einer Zelle Z^k ($k \leq m-1$), S^{k-1} , S^{k-2} , E_i^{k-1} dieselbe Bedeutung wie vorher, so bilden die den Zellen von S^{k-1} , S^{k-2} , E_i^{k-1} entsprechenden Zellen von S^m bzw. E^m zusammen mit Z_1^0 und Z_2^0 nach Satz 6 ($k-1$) und 6 ($k-2$) je eine S^k , S^{k-1} , E_i^k ; daher bewirkt die i. T. von S^{m-1} bzw. E^{m-1} eine i. T. von S^m bzw. E^m , wodurch diese wieder in eine S^m bzw. E^m übergeht. Dasselbe gilt bei Vereinigung zweier Zellen.

Damit sind endlich die Sätze 2 bis 6 bewiesen.

Wir brauchen noch den folgenden

Satz 7. Jede i. T. eines $UC_{Z^k}(C^n)$ lässt sich durch eine i. T. von C^n bewirken. Ist $k=0$, und besteht die i. T. von UC_{Z^k} nur in Teilungen, so lässt sie sich bewirken durch Teilungen in C^n , bei denen keine von Z^0 berandete Strecke betroffen wird, wenn nur anfangs keine zwei Strecken beide Endpunkte gemeinsam haben.

Wir teilen jede von Z^k berandete Zelle in folgender Weise. Jede Z^{k+1} wird geteilt durch eine Zelle \bar{Z}^k , die von derselben S^{k-1} berandet wird wie Z^k . (Diesen Schritt unterlassen wir nur, wenn es sich im Falle $k=0$ darum handelt, die zweite Behauptung zu beweisen). Haben wir alle von Z^k berandeten Zellen bis herauf zur l -ten Dimension geteilt, so teilen wir jede von Z^k berandete Z^{l+1}

durch eine Zelle \bar{Z}^l , deren Rand besteht aus S^{k-1} und allen den Zellen \bar{Z}^r , deren entsprechende Z^{r+1} Randzellen von Z^{l+1} sind. Bei diesen Teilungen hat sich UC_{Z^k} nicht geändert. Wollen wir nun in UC_{Z^k} eine i.T. ausführen, etwa Z^{l-k-1} , das Bild von Z^l , teilen durch eine Z^{l-k-2} , deren Rand eine auf der Rand- S^{l-k-2} von Z^{l-k-1} gelegene und diese in zwei E^{l-k-2} zerlegende S^{l-k-3} ist, so teilen wir \bar{Z}^{l-1} durch eine \bar{Z}^{l-2} , deren Rand besteht aus S^k und denjenigen \bar{Z}^r , deren zugehörigen Z^{r+1} auf UC_{Z^k} die Zellen von S^{l-k-3} entsprechen, darauf teilen wir Z^l durch eine Z^{l-1} , deren Rand besteht aus den Zellen: \bar{Z}^{l-2} mit ihren Randzellen (\bar{S}^{l-3}), Z^k mit ihren Randzellen (S^{k-1}) und zu jeder \bar{Z}^r von S^{l-3} die zugehörige \bar{Z}^r . Auf UC_{Z^k} wird damit Z^{l-k-1} geteilt durch Z^{l-k-2} , das Bild von Z^{l-1} , und der Rand von Z^{l-k-2} ist die vorgegebene S^{l-k-3} . Sollen zwei Zellen Z_1^{l-k-1} und Z_2^{l-k-1} von UC_{Z^k} durch Beseitigung der gemeinsamen Randzelle Z^{l-k-2} vereinigt werden, so vereinigen wir die entsprechenden Zellen Z_1^l und Z_2^l von C^n durch Beseitigung der gemeinsamen Randzelle Z^{l-1} . Soll eine weitere i.T. auf UC_{Z^k} bewirkt werden, so wiederholen wir das ganze Verfahren. Die Rechtmässigkeit der vorgenommenen i.T. folgt aus den Sätzen:

(A) Lassen wir von einer S^n einen E^n weg, so bleibt ein E^n übrig.

(B) Erhöhen wir in einer S^n (einem E^n) die Dimension jeder Zelle um k und fügen wir eine S^{k-1} hinzu, deren Zellen alle anderen Zellen beranden sollen, so erhalten wir eine S^{n+k} (einen E^{n+k}).

Der Beweis von (A) ist nicht ganz einfach, aber von den folgenden Betrachtungen unabhängig. Ich denke ihn in anderem Zusammenhange zu geben.

Um (B) zu beweisen, transformieren wir S^{k-1} in die Normalgestalt. Das ist durch eine Transformation des ganzen Komplexes möglich, da alle Zellen von S^{k-1} dieselben Zellen, nämlich sämtliche, die nicht zu S^{k-1} gehören, beranden. Hat aber S^{k-1} die Normalgestalt, so folgt (B) durch $(k+1)$ -malige Anwendung des Sonderfalles $k=0$, und dieser wurde schon als Satz 6 bewiesen.

Mit Hilfe des Begriffes UC können wir jetzt definieren:

Eine geschlossene n -dimensionale ($n > \sigma$) Mannigfaltigkeit M^n ist ein zusammenhängender C^n , in dem jeder UC_{Z^0} eine S^{n-1} ist. Nach (I) und Satz 3($n-1$) ist dann auch jeder UC_{Z^k} eine S^{n-k-1} . Genau wie beim Beweis der Sätze 3 bis 5 zeigt sich, dass eine M^n bei

i.T. wieder in eine M^n übergeht, dass der Begriff Mannigfaltigkeit gegenüber Homöomorphie invariant ist.

§ 2. Zerlegung in höchstens zwei Teile.

Satz 8(n). *Liegt in einer M^n ($n \geq 1$) eine M^{n-1} , so lassen sich ein beliebiger Punkt Q von M^{n-1} und ein beliebiger nicht auf M^{n-1} liegender Punkt P nach einer geeigneten Unterteilung von M^n durch einem M^{n-1} nur in Q treffenden Streckenzug verbinden.*

Satz 9(n). *Eine M^n wird durch eine in ihr liegende M^{n-1} in höchstens zwei Teile zerlegt, d.h. von drei beliebigen nicht auf M^{n-1} liegenden Punkten von M^n lassen sich nach einer geeigneten Unterteilung von M^n mindestens zwei durch einen M^{n-1} nicht treffenden Streckenzug verbinden.*

Offenbar gelten die Sätze 8(1) und 9(1), wenn wir, wie es hier zweckmässig ist, unter M^0 nur die S^0 , d.h. zwei Punkte verstehen. Es seien die Sätze 8($n-1$) und 9($n-1$) bewiesen; wir beweisen zuerst 8(n), dann 9(n).

Da M^n ein zusammenhängender C^n ist, können wir P und Q durch einen Streckenzug verbinden. Ist Q_1 der erste auf M^{n-1} liegende Punkt des Streckenzuges, und fällt Q_1 mit Q zusammen, so ist Satz 8 bestätigt. Anderenfalls können wir, da M^{n-1} zusammenhängt, Q_1 und Q durch einen auf M^{n-1} verlaufenden Streckenzug $Q_1 Q_2 \dots Q_r Q$ verbinden. Von diesem dürfen wir annehmen, dass er keinen Doppelpunkt hat; denn sonst könnten wir ihn durch Weglassen geschlossener Streckenzüge verkürzen. Wir ersetzen den Streckenzug $PQ_1 Q_2 \dots Q_r Q$ durch einen Zug $PQ_2 \dots Q_r Q$, dessen Teil PQ_2 die Mannigfaltigkeit M^{n-1} nur in Q_2 trifft. Das geschieht folgendermassen. Auf $UC_{Q_1}(M^n) = S^{n-1}$ ist das Bild von M^{n-1} eine S^{n-2} , also eine besondere M^{n-2} , das der Strecke $Q_1 Q_2$ ein Punkt von S^{n-2} , das der letzten Strecke RQ_1 des Zuges PQ_1 ein nicht auf S^{n-2} liegender Punkt. Nach Satz 8($n-1$), angewandt auf S^{n-1} , können wir diese beiden Punkte nach einer geeigneten Unterteilung von S^{n-1} durch einen S^{n-2} nicht treffenden Streckenzug auf S^{n-1} verbinden. Diese Unterteilung von S^{n-1} lässt sich nach Satz 7 bewirken durch eine Unterteilung von M^n , bei der die Strecke $Q_1 Q_2$ nicht geteilt wird. Das bedeutet aber für M^n , dass nach dieser Unterteilung eine Folge von Flächenzellen $Z_1^2, Z_2^2, \dots, Z_s^2$

vorliegt, die sämtlich in Q_1 zusammenstossen, von denen die erste von RQ_1 und die letzte von $Q_1 Q_s$ berandet wird, und von denen jede $Z_i^2 (i=1, \dots, s-1)$ mit der folgenden, Z_{i+1}^2 eine von Q_1 ausgehende, nicht auf M^{n-1} liegende Strecke Z_i^1 gemeinsam hat. Jede dieser gemeinsamen Randstrecken Z_i^1 teilen wir durch einen neu eingeführten Punkt T_i und verbinden R mit T_1 durch eine Strecke auf Z_1^2 , T_i mit T_{i+1} ($i=1, \dots, s-1$) durch eine Strecke auf Z_{i+1}^2 , T_{s-1} mit Q_s durch eine Strecke auf Z_s^2 . Schliesslich ersetzen wir die Strecken $RQ_1 Q_s$ des Zuges PQ durch die Strecken $RT_1 T_2 \dots T_{s-1} Q_s$; damit haben wir die Anzahl der auf M^{n-1} liegenden Strecken des Zuges um eins vermindert. Wiederholung dieses Verfahrens liefert zuletzt einen Streckenzug PQ , der M^{n-1} nur in Q trifft, wie es Satz 8 behauptet.

Sind nun P_1, P_2, P_3 drei nicht auf M^{n-1} liegende Punkte von M^n , so verbinden wir jeden von ihnen mit einem Punkt Q von M^{n-1} durch einen M^{n-1} nur in Q treffenden Streckenzug, was nach Satz 8 (n) nach einer Unterteilung von M^n möglich ist. Sind $R_1 Q, R_2 Q, R_3 Q$ die letzten Strecken dieser Züge, so sind deren Bilder auf $UCQ = S^{n-1}$ drei Punkte, die nicht auf dem Bilde S^{n-2} von M^{n-1} liegen. Nach Satz 9 ($n-1$), angewandt auf S^{n-2} und die Bilder von $R_1 Q, R_2 Q$ und $R_3 Q$ auf S^{n-1} , lassen sich nach einer Unterteilung von S^{n-1} zwei von den drei Punkten, durch einen S^{n-2} nicht treffenden Streckenzug verbinden. Nach Satz 7 lässt sich diese Unterteilung von S^{n-1} durch eine Unterteilung von M^n bewirken. Nach dieser Unterteilung haben wir eine Folge in Q zusammenstossender Flächenzellen $Z_1^2, Z_2^2, \dots, Z_s^2$, die nicht auf M^{n-1} liegen von denen etwa die erste von $R_1 Q$, die letzte von $R_s Q$ berandet wird, und von denen jede, Z_i^2 mit der folgenden Z_{i+1}^2 in einer nicht auf M^{n-1} liegenden Strecke Z_i^1 zusammenstösst. Auf Z_i^1 führen wir je einen neuen Punkt T_i ein und verbinden R_1 mit T_1 , T_i mit T_{i+1} , T_{s-1} mit R_s , durch je eine Strecke innerhalb der Zellen Z_1^2, \dots, Z_s^2 . Damit sind R_1 und R_s , also P_1 und P_s durch einen M^{n-1} nicht treffenden Streckenzug verbunden, wie es Satz 9 (n) behauptet.

§ 3. Zerlegung in genau zwei Teile.

Satz 10(n). Ist bei der Mannigfaltigkeit M^n ($n > 1$) die erste

BETTISCHE Zahl $P_1 = 1$, und sind alle Torsionszahlen erster Dimension ungerade, so wird M^n durch jede M^{n-1} zerlegt, d.h. wir können durch eine Unterteilung von M^n zwei Punkte einführen, die sich durch keinen M^{n-1} nicht treffenden Streckenzug verbinden lassen.

Satz 10(m) sei bewiesen für $m < n$. Die Sphäre S^m erfüllt die Voraussetzungen, wenn $m > 1$ ist: es ist $P_1 = 1$ und keine Torsionszahl vorhanden. Im Falle $m = 1$ gilt die Behauptung von Satz 10; denn eine M^1 , d.h. eine geschlossene Kurve wird durch jede M^0 , d.h. durch irgend zwei Punkte zerlegt (obwohl hier $P_1 = 2$ ist).

Wir wählen einen beliebigen Punkt Q auf M^{n-1} . UC_Q ist eine S^{n-1} , das Bild von M^{n-1} auf UC_Q eine S^{n-2} . Nach Satz 10($n-1$) können wir S^{n-1} so unterteilen, dass zwei nicht auf S^{n-2} liegende Punkte O_1 und O_2 durch S^{n-2} getrennt werden. Nach Satz 7 können wir die Unterteilung von S^{n-1} durch eine Unterteilung von M^n bewirken. Dann teilen wir die beiden von Q ausgehenden Strecken, deren Bilder O_1 und O_2 sind, durch die Punkte P_1 und P_2 und behaupten: P_1 und P_2 lassen sich durch keinen M^{n-1} nicht treffenden Streckenzug verbinden. Liessen sie sich nämlich verbinden, so würde der verbindende Streckenzug mit den Strecken P_1QP_2 eine geschlossene Kurve bilden, deren Lage in ihrem einzigen Schnittpunkt Q mit M^{n-1} durch die Konstruktion so eingerichtet ist, dass wir in sinngemässer Übertragung der üblichen Ausdrucksweise sagen können, sie durchschneidet M^{n-1} in Q . Unter den Voraussetzungen von Satz 10 ist jede geschlossene Kurve G^1 , eine geeignete ungerade Anzahl von Malen genommen, homolog Null, d.h. der vollständige Rand eines Aggregats G^2 orientierter Flächenzellen. Von der Orientierung können wir für unseren Zweck absehen und merken nur an, dass in den Strecken von G^1 eine ungerade, in allen anderen Strecken eine gerade Zahl Zellen von G^2 angrenzen. Liegen nun (wenn $n > 2$ ist) Flächenzellen von G^2 auf M^{n-1} , so ersetzen wir sie durch andere. Ist nämlich Z^2 eine solche, so können wir durch eine Unterteilung von M^n bewirken, dass an Z^2 eine nicht zu M^{n-1} gehörende Z^2 anstösst. Dann teilen wir Z^2 durch eine Flächenzelle Z_1^2 , die wir in die Randkurve von Z^2 einspannen, und ersetzen Z^2 als Zelle von G^2 durch Z_1^2 .

Ist Z^1 eine auf M^{n-1} liegende Strecke von G^2 und $n > 2$, so ist $UC_{Z^1}(M^n)$ eine S^{n-2} , das Bild von M^{n-1} auf $UC_{Z^1}(M^n)$ eine S^{n-3} , das von G^2 eine gerade Anzahl nicht auf S^{n-3} liegender

Punkte. Nach Satz 9($n-2$) und 10($n-2$) wird S^{n-2} durch S^{n-3} in zwei Teile zerlegt. Z^1 heie Schnittstrecke, wenn in jedem der beiden Teile von S^{n-2} eine ungerade Anzahl dieser Punkte liegt. Ist $n=2$ so besteht UC_{Z^1} aus zwei Punkten und das Bild von G^2 aus einer geraden Anzahl von Punkten. Z^1 heie Schnittstrecke, wenn je eine ungerade Anzahl mit jedem der beiden Punkte von UC_{Z^1} zusammenfllt.

Ist P ein auf M^{n-1} liegender Punkt von G^2 , so ist $UC_P(M^n)$ eine S^{n-1} , das Bild von M^{n-1} auf UC_P eine S_1^{n-2} , das von G^2 ein Streckenkomplex C^1 , in dessen Punkten immer eine gerade Zahl von Strecken zusammenstsst. Den auf M^{n-1} liegenden Strecken von G^2 entsprechen auf S_1^{n-2} liegende Punkte von C^1 . Von einem solchen Punkt R fhren in jeden der beiden Teile, in die S^{n-1} nach Satz 8($n-1$) durch S_1^{n-2} zerlegt wird, eine Anzahl Strecken. Die dem Punkt R entsprechende Strecke von G^2 ist dann und nur dann Schnittstrecke, wenn diese Anzahlen beide ungerade sind. Den Punkten von $UC_{Z^1}(M^n)$, die in demselben von S^{n-3} bestimmten Teil liegen, entsprechen auf $UC_P(M^n)$ Strecken, die sich nach einer Unterteilung durch eine Folge nicht auf S^{n-2} liegender Strecken und Flchenzellen verbinden lassen, also zu demselben durch S^{n-1} bestimmten Teil von $UC_P(M^n)$ gehren. Wir behaupten nun: in Q endigt eine ungerade, in jedem anderen auf M^n liegenden Punkt P von G^2 eine gerade Zahl Schnittstrecken. Auf $UC_P(P \neq Q)$ haben wir nmlich einen Streckenkomplex, in dessen Punkten je eine gerade Anzahl Strecken endigt. Dieser lsst sich in eine Anzahl geschlossener Kurven zerlegen, und von diesen muss jede den einen durch S_1^{n-1} auf UC_P bestimmten Teil gleich oft betreten und verlassen, also S_1^{n-2} im ganzen eine gerade Anzahl Male schneiden. Auf UC_Q dagegen haben wir einen Streckenkomplex, in dessen Punkten je eine gerade Zahl Strecken endigt, mit Ausnahme der Punkte O_1 und O_2 , die in verschiedenen durch S_1^{n-2} bestimmten Teilen von UC_Q liegen, und in denen je eine ungerade Zahl Strecken endigt. Der Streckenkomplex lsst sich zerlegen in geschlossene Kurven und eine O_1 mit O_2 verbindende Kurve. Die ersteren schneiden S_1^{n-2} eine gerade, die letzte eine ungerade Anzahl Male.

Zhlen wir nun die Endpunkte aller Schnittstrecken, so liefert Q einen ungeraden, jeder andere Punkt einen geraden Beitrag. Die Gesamtzahl ist aber gerade, da jede Strecke zwei Endpunkte hat.

Also ist die Annahme, M^n würde durch M^{n-1} nicht zerlegt, zu verwerfen, Satz 10(n) ist bewiesen.

§ 4. Nicht zerlegende Mannigfaltigkeiten.

Satz 11. Ist die erste BETTISCHE Zahl der Mannigfaltigkeit M^n ($n > 1$) grösser als eins, oder ist eine gerade Torsionszahl erster Dimension vorhanden, so gibt es eine M^{n-1} auf M^n , die M^n nicht zerlegt.

Um die Voraussetzung auszunutzen, erinnern wir uns an die Bedeutung der BETTISCHEN und TORSIONSZAHLEN. Ist P_1 die erste BETTISCHE Zahl, und sind τ_1, \dots, τ_r die TORSIONSZAHLEN erster Dimension, so gibt es eine Basis für die gerichteten geschlossenen Wege auf M^n , bestehend aus den Wegen $X_1, \dots, X_{P_1-1}, Y_1, \dots, Y_r$ derart, dass sich jeder geschlossene gerichtete Weg X bis auf Homologie aus ihnen kombinieren lässt:

$$X \sim \sum_{v=1}^{P_1-1} c_v X_v + \sum_{v=1}^r d_v Y_v, \quad (1)$$

und dann und nur dann $X \sim 0$ ist, wenn

$$c_v = 0 \quad (v = 1, \dots, P_1-1), \quad d_v \equiv 0 \pmod{\tau_v} \quad (v = 1, \dots, r)$$

ist.

Statt der Wege, d.h. der geschlossenen gerichteten Ketten von Zellen abwechselnd nullter und erster Dimension, von denen je zwei aufeinander folgende in der Berandungsbeziehung stehen, betrachten wir jetzt ebensolche Ketten von Zellen abwechselnd n -ter und $(n-1)$ -ter Dimension. Jeder solchen Kette

$$Z_1^n Z_1^{n-1} Z_2^n \dots Z_r^{n-1} Z_1^n \quad (2)$$

ordnen wir einen Weg zu, indem wir auf dem Rande jeder Zelle Z_i^{n-1} einen Punkt P_i bestimmen und jeweils P_i mit P_{i+1} ($i=1, \dots, r-1$) durch einen Weg auf dem Rande von Z_{i+1}^n , P_r mit P_1 durch einen Weg auf dem Rande von Z_1^n verbinden. Zwei verschiedene, derselben Kette zugeordnete Wege sind einander homolog. Denn ist $P'_1 P'_2 \dots P'_r P'_1$ ein anderer der Kette (2) zugeordneter Weg, so verbinden wir jeweils P_i und P'_i durch einen Weg auf dem Rande von Z_i^{n-1} . Da der Rand von Z_i^n eine S^{n-1} ist, so ist der auf ihm gelegene geschlossene Weg $P_{i-1} P_i P'_i P'_{i-1} P_{i-1}$ bzw. für $i=1$ der Weg $P_r P_1 P'_1 P'_r P_r$ homolog Null. Die Addition dieser Homologien ergibt

$$P_1 P_2 \dots P_r P_1 \sim P'_1 P'_2 \dots P'_r P'_1.$$

Wir können also auf die Ketten (2) die Basisdarstellung (1) anwenden. Umgekehrt gehört auch zu jedem Weg $Z_1^0 Z_1^1 Z_2^0 \dots Z_s^1 Z_1^0$ eine Kette (2). Wir brauchen nur jeder Strecke Z_i^1 des Weges eine von ihr berandete Zelle Z_i^n zuzuordnen und Z_i^n mit Z_{i+1}^n ($i=1, \dots, s-1$) durch eine Kette von Z_{i+1}^0 berandeter, Z_s^n mit Z_1^n durch eine Kette von Z_1^0 berandeter Zellen n -ter und $(n-1)$ -ter Dimension zu verbinden. Dies ist stets möglich, da die von Z_i^0 berandeten Zellen die Bilder der Zellen von $UC_{Z_i^0}$, einer S^{n-1} sind, und man auf dieser je zwei Zellen $(n-1)$ -ter Dimension durch eine Kette von Zellen $(n-1)$ -ter und $(n-2)$ -ter Dimension verbinden kann. Der so gefundenen Kette kann offenbar nach der oben angegebenen Regel der ursprüngliche Weg, vermehrt um hin und zurück durchlaufene Strecken, also ein dem ursprünglichen homologer Weg, zugeordnet werden.

Der UC einer Z^{n-2} ist eine S^1 . Umlaufen wir sie, so erhalten wir eine Kette, der ein verschwindender Weg zugeordnet werden kann: wir brauchen nur für alle Punkte P_i denselben Randpunkt von Z^{n-2} zu nehmen.

Mit Hilfe der Basisdarstellung (1) für die Ketten (2) konstruieren wir einen Komplex $(n-1)$ -ter Dimension C^{n-1} , aus dem wir dann eine M^n nicht zerlegende M^{n-1} ableiten werden.

Wir wählen eine Zelle n -ter Dimension Z_1^n . An sie grenze in der Zelle Z_1^{n-1} eine andere, Z_2^n . Diese nehmen wir zu Z_1^n hinzu und setzen fest, dass Z_1^{n-1} nicht zu C^{n-1} gehören soll. Haben wir so die Zellen Z_1^n, \dots, Z_r^n bekommen, und gibt es noch weitere Zellen n -ter Dimension, so sei unter diesen Z_{r+1}^n eine, die in der Zelle Z_r^{n-1} an eine der Zellen Z_1^n, \dots, Z_r^n angrenzt. Wir nehmen Z_{r+1}^n hinzu und setzen fest, dass Z_r^{n-1} nicht zu C^{n-1} gehört. Haben wir so alle Z^n erschöpft, so gilt es noch, für die bisher nicht aufgetretenen Z^{n-1} festzusetzen, ob sie zu C^{n-1} gehören oder nicht. Z^{n-1} berande die beiden Zellen Z_k^n und Z_l^n . Jede von diesen ist mit Z_1^n durch eine Kette von Zellen n -ter und $(n-1)$ -ter Dimension verbunden, deren Zellen $(n-1)$ -ter Dimension nicht zu C^{n-1} gehören. Ist $P_1 > 1$ und c_1 der Wert aus der Basisdarstellung (1) der Kette $Z_1^n \dots Z_k^n Z^{n-1} Z_l^n \dots Z_1^n$, so gehöre Z^{n-1} zu C^{n-1} wenn c_1 ungerade ist, sonst nicht. Ist aber $P_1 = 1$ und eine der Torsionszahlen, etwa τ_1 , gerade, und d_1 wieder der

Wert aus der Basisdarstellung (1) der Kette $Z_1^n \dots Z_k^n Z^{n-1} Z_l^n \dots Z_1^n$ so gehöre Z^{n-1} zu C^{n-1} wenn d_1 ungerade ist, sonst nicht. Da d_1 bis auf ein Vielfaches der geraden Zahl τ_1 bestimmt ist, ist damit die Zugehörigkeit eindeutig festgelegt.

Der Komplex C^{n-1} hat die folgenden Eigenschaften:

1. C^{n-1} ist nicht leer. Wählen wir nämlich eine dem Weg X_1 bzw. Y_1 zugeordnete Kette $Z_{i_1}^n Z_{k_1}^{n-1} Z_{i_2}^n \dots Z_{k_r}^{n-1} Z_{i_l}^n$, und ergänzen wir sie zu der Kette $Z_1^n \dots Z_{i_1}^n Z_{k_1}^{n-1} Z_{i_2}^n \dots Z_1^n \dots Z_{k_r}^{n-1} Z_{i_l}^n \dots Z_1^n$, in der bei jeder Zelle $Z_{i_l}^n$ die keine Zelle von C^{n-1} enthaltende Kette $Z_{i_l}^n \dots Z_1^n \dots Z_{i_l}^n$ eingeschaltet ist, so enthält die ergänzte Kette dieselben Zellen von C^{n-1} wie die ursprüngliche. Wäre dies keine, so wären in der Darstellung (1) der Ketten $Z_1^n \dots Z_{i_l}^n Z_{k_l}^{n-1} Z_{i_{l+1}}^n \dots Z_1^n$ immer c_1 bzw. d_1 gerade. In der Darstellung (1) der gesamten Kette hat aber c_1 bzw. d_1 den Wert 1, und dieser ist (mod. τ_1) die Summe jener.

2. C^{n-1} zerlegt M^n nicht; denn wir haben ja jede Z^n mit Z_1^n durch eine keine Zelle von C^{n-1} enthaltende Kette verbunden.

3. In jeder Z^{n-2} von M^n stösst eine gerade Anzahl Zellen von C^{n-1} zusammen. Denn umkreisen wir Z^{n-2} durch eine Kette, so ist jeder zugehörige Weg homolog Null, insbesondere in der Basisdarstellung $c_1 = 0$ bzw. $d_1 \equiv 0$; genau wie unter 1. folgt daraus die Behauptung.

Vermöge dieser Eigenschaften gelingt es, die Singularitäten von C^{n-1} aufzulösen, d.h. C^{n-1} so abzuändern, dass sich schliesslich eine M^n nicht zerlegende M^{n-1} ergibt. Eine Zelle Z^k von C^{n-1} , d.h. eine Randzelle einer Z^{n-1} von C^{n-1} , heisse regulär, wenn $UC_{Z^k}(C^{n-1})$ eine S^{n-k-2} ist, die $UC_{Z^k}(M^n)$, eine S^{n-k-1} , in zwei E^{n-k-1} zerlegt. Eine Z^{n-2} ist also regulär, wenn in ihr genau zwei Z^{n-1} von C^{n-1} zusammenstossen. Ist dies nicht der Fall, ist die Anzahl der an Z^{n-2} anstossenden Z^{n-1} von C^{n-1} (gerade und) grösser als zwei, so üben wir zunächst auf die an Z^{n-2} anstossenden Zellen von M^n die beim Beweis von Satz 7 vorgenommenen Teilungen aus. Danach entspricht jeder von Z^{n-2} berandeten Z^l ($l=n-1, n$) eine sie berandende \bar{Z}^{l-1} . Zwei benachbarte Z^{n-1} von C^{n-1} , d.h.

solche, zwischen denen im Sinne der zyklischen Ordnung auf $UC_{Z^{n-2}}$ keine weitere Z^{n-1} von C^{n-1} liegt, nehmen wir von C^{n-1} weg und ersetzen sie durch diejenigen \bar{Z}^{n-1} , die den ihre Bildpunkte auf $UC_{Z^{n-2}}$ verbindenden Strecken entsprechen. Die hierbei neu auftretenden Z^{n-2} von C^{n-1} sind regulär, und die Anzahl der in Z^{n-2} anstossenden Z^{n-1} von C^{n-1} ist um zwei vermindert worden. Daher können durch wiederholte Anwendung dieses Verfahrens schliesslich alle singulären Z^{n-2} aus C^{n-1} beseitigt werden. Auch der abgeänderte C^{n-1} zerlegt M^n nicht; denn betrachten wir die Teilung so, als hätten wir von den an Z^{n-2} anstossenden Zellen kleine Stücke abgetrennt, so wird der Rest sicher nicht von C^{n-1} zerlegt; von den abgetrennten Zellen lässt sich aber jede mit einer Restzelle durch eine C^{n-1} nicht treffende Kette verbinden.

Es seien nun alle Zellen von höherer als k -ter Dimension von C^{n-1} regulär. Gehört die Zelle Z^k zu C^{n-1} , so ist $UC_{Z^k}(C^{n-1})$ ein Komplex C^{n-k-2} auf $UC_{Z^k}(M^n) = S^{n-k-1}$, und wir behaupten, dass dieser aus einer Anzahl M^{n-k-2} besteht. Wir brauchen nur zu beweisen, dass der UC jedes Punktes eine S^{n-k-3} ist. Der Punkt Z^0 sei das Bild der Zelle Z^{k+1} von C^{n-1} . Dann ist nach (I)

$$UC_{Z^0}(UC_{Z^k}(C^{n-1})) = UC_{Z^{k+1}}(C^{n-1}) = S^{n-k-3}.$$

Jeder grösste zusammenhängende Teilkomplex von $UC_{Z^k}(C^{n-1})$ ist also eine M^{n-k-2} , d. h. $UC_{Z^k}(C^{n-1})$ zerfällt in eine Anzahl M^{n-k-2} . Unter diesen suchen wir eine „innerste“, d. h. eine von der Art, dass in dem einen der beiden Teile, in die sie (nach Satz 10) $UC_{Z^k}(M^n)$ zerlegt, keine andere Teilmannigfaltigkeit von $UC_{Z^k}(C^{n-1})$ enthalten ist. Ist nämlich Q eine nicht zu $UC_{Z^k}(C^{n-1})$ gehörige Zelle von $UC_{Z^k}(M^n)$, und bezeichnen wir als Inneres einer Teilmannigfaltigkeit M denjenigen durch M bestimmten Teil von $UC_{Z^k}(M^n)$, der Q nicht enthält, so brauchen wir nur von einer Teilmannigfaltigkeit zu einer in ihrem Inneren enthaltenen überzugehen, bis keine solche mehr vorhanden ist, bis wir eine innerste M^{n-k-2} gefunden haben. Jetzt führen wir mit den von Z^k berandeten Zellen die Teilungen aus dem Beweis von Satz 7 aus und ersetzen die den

Zellen von M^{n-k-2} entsprechenden Zellen Z^i von C^{n-1} durch diejenigen Zellen \bar{Z}^{i-1} , die die den Zellen des Inneren von M^{n-k-2} entsprechenden Zellen beranden. Auch nach dieser Abänderung wird M^n nicht durch C^{n-1} zerlegt; denn die dem Inneren von M^{n-k-2} entsprechenden von Z^k berandeten Zellen lassen sich nunmehr mit anderen von Z^k berandeten Zellen durch Ketten verbinden, die C^{n-1} nicht überschreiten.

Wir haben noch zu zeigen, dass jede der neu eingeführten Zellen k -ter Dimension \bar{Z}^k von C^{n-1} regulär ist. Durch eine solche Zelle wurde eine nach Voraussetzung reguläre Zelle Z^{k+1} von C^{n-1} geteilt. Ihr UC entsteht also, indem wir in $UC_{Z^{k+1}}$ die Dimension jeder Zelle um eins erhöhen und zwei Punkte hinzufügen. Nach Satz 6 entsteht so aus $UC_{Z^{k+1}}(C^{n-1}) = S^{n-k-3}$ eine S^{n-k-2} und aus jedem der E^{n-k-3} , in die $UC_{Z^{k+1}}(M^n)$ durch $UC_{Z^{k+1}}(C^{n-1})$ zerlegt wird, ein E^{n-k-2} . Durch die folgenden Teilungen wird noch jede Zelle von $UC_{\bar{Z}^k}(M^n)$ geteilt, sodass schliesslich eine in der Art der Oberfläche einer Doppelpyramide geteilte S^{n-k+1} vorliegt: sie besteht aus einer S^{n-k-2} (das ist $UC_{Z^{k+1}}(M^n)$), zwei Punkten P_1 und P_2 , den „Spitzen“, und zu jeder Zelle von S^{n-k-2} zwei Zellen der nächst höheren Dimension. Nach der Abänderung bildet $UC_{\bar{Z}^k}(C^{n-1})$ auf dieser S^{n-k-1} einen Komplex der folgenden Art: es gehören dazu die Zellen eines auf S^{n-k-2} gelegenen E^{n-k-1} (nämlich eines der beiden E^{n-k-1} , in die $UC_{Z^{k+1}}(M^n)$ durch $UC_{Z^{k+1}}(C^{n-1})$ zerlegt wird) und diejenigen Zellen, die diese mit einer der beiden Spitzen der Doppelpyramide, etwa mit P_1 , verbinden. Dass der eine (und daher auch der andere) hierdurch bestimmte Teil von $UC_{\bar{Z}^k}(M^n)$, nämlich die Gesamtheit der E^{n-k-2} mit P_1 verbindenden Zellen ein E^{n-k-1} ist, ist nicht schwer zu beweisen und soll an anderer Stelle gezeigt werden. Damit ist bewiesen, dass \bar{Z}^k eine reguläre Zelle ist. Jetzt besteht aber $UC_{\bar{Z}^k}(C^{n-1})$ aus einer M^{n-k-2} weniger als vorher, und wir können das Verfahren wiederholen, bis Z^k überhaupt nicht mehr zu C^{n-1} gehört, ohne dabei singuläre Z^k hinzuzubekommen. Dies Verfahren wenden wir nacheinander auf alle Zellen $(n-2)$ -ter bis nullter Dimension an und erhalten zum Schluss C^{n-1} in einer

Gestalt mit lauter regulären Zellen, sodass insbesondere jeder UC eines Punktes eine S^{n-2} ist. Es zerfällt also C^{n-1} in eine Anzahl M^{n-1} . Da diese alle zusammen M^n nicht zerlegen, tut es eine von ihnen allein erst recht nicht: wir haben eine M^n nicht zerlegende M^{n-1} , wie es Satz 11 behauptet.

Die Sätze 9, 10 und 11 ergeben zusammen Satz 1.

Physics. — “*Preliminary measurements concerning the dielectric constants of liquid hydrogen and liquid oxygen and its dependence on temperature as regards the latter*”. By G. BREIT and H. KAMERLINGH ONNES. Communication N°. 171a from the Physical Laboratory at Leiden.

(Communicated at the meeting of September 27, 1924).

§ 1. *Introduction.* The following measurements concerning the dielectric constants of liquefied gases are the first performed again at Leiden after those of HASENÖHRL.¹⁾ Such determinations had remained on the program indeed and had even obtained a large interest, when a maximum of density had been found for liquid helium, but they had to be neglected again and again for more urgent problems. We also, when we got the opportunity to take at hand the determinations mentioned by means of high frequency oscillations — HASENÖHRL used low frequency — could perform this research as second place work only. Further, the departure of one of us (G. B.) caused it to be stopped before the certainty and the accuracy of the measurements had been increased to the required degree. So we cannot but consider our results as preliminary ones only. Yet the first determination of a so interesting quantity as the dielectric constant of liquid hydrogen seems to deserve to be published. The same holds for the observations showing that within the limit of their (though small) accuracy the change of the dielectric constant of liquid oxygen with temperature down from the boiling point can be expressed by the formula of CLAUSIUS-MOSOTTI.

As regards liquid hydrogen it could only be stated, that the change in dielectric constant is of the same order as indicated by the formula mentioned.

§ 2. *Method and apparatus.* We used sustained oscillations of a frequency of the order of 100,000 produced by means of triode valves. The circuit of the triode valve was coupled loosely to the circuit of the condenser placed in the cryostat and filled with liquefied gas.

¹⁾ These Proceedings II (1899), p. 211, Leiden Comm. N°. 52.

Two procedures have been employed:

1. We substituted for the experimental condenser a calibrated measuring condenser that could be adjusted to the required capacity (substitution method).

2. We connected the experimental condenser in parallel with a calibrated measuring condenser of larger capacity which had been first adjusted to resonance and readjusted it again to resonance after the connexion had been made. The difference of the two settings gives the capacity of the experimental condenser plus that of the leads (parallel connexion method).

The measuring condenser (capacity about 460 $\mu\mu f$) consists of two sets of zinc plates supported each by a set of three vertical brass columns (six columns in all). The columns were mounted in two ebonite rings. The whole condenser was shielded electrostatically by means of a cylinder of german silver with a number of perforations allowing a free circulation of the liquid. The leads to the condenser were of bare copper and were insulated from the cap of the cryostat by means of glass tubing and KOTHIISKY cement. In the silvering of the vacuumglasses of the cryostats vertical slits had been left unsilvered allowing one to follow the level of the liquid.

The measurements were performed in the following order:

1. The cryostat was evacuated and the capacity of the condenser and leads was measured when at atmospheric temperature;

2. the cryostat was cooled and the capacity of the condenser and leads was measured;

3. the cryostat was filled with the liquefied gas, the pressure of evaporation was kept constant, the level of the liquid was noted and the capacity of the condenser was measured;

4. the pressure in the cryostat was then reduced, the level of the liquid noted and the capacity measured again.

In observations at different temperatures the pressure had to be always used first at high and then at low values so as to provide for proper circulation of the liquid.

The experimental condenser was calibrated at ordinary temperatures at the Bureau of Standards in Washington D. C., U. S. A. and the Croft Laboratory of the Harvard University and also at Leiden by comparison with a condenser which has been later calibrated in Washington. The correction for the leads has been ascertained by special trial measurements.

§ 3. *Dielectric constant of liquid hydrogen.* The measurements have been performed on May 24, May 26 and June 24, 1922. As

in the later measurements use could be made of the experience gained in the earlier ones, those of June 24 are the most reliable. In those of May 24 the cap of the cryostat had not yet been earthed. However, in table I we give all the results. The dielectric constant K is given as compared with that of vacuum, the pressure of the evaporating gas in international cms mercury, the temperature on the provisional international KELVIN scale. In the last column the ratio $\frac{K-1}{K+2} \cdot \frac{1}{D}$, which according to the formula of CLAUSIUS-MOSOTTI had to be a constant is given. The latter column cannot show much more than that the change in K is of the same order of magnitude as required by the formula of CLAUSIUS-MOSOTTI.

TABLE I.

| Liquid hydrogen | | | | | | |
|-----------------|---------|----------|---------------------------|-----------------------|-------|-------------------------------------|
| Date | Method | Pressure | Temperature ¹⁾ | Density ²⁾ | K | $\frac{K-1}{K+2} \cdot \frac{1}{D}$ |
| 24 May 1922 | Subst. | 76.3 cm | 20°.37 K | 0.0708 | 1.211 | 0.928 |
| 26 " " | Parall. | 76.7 | 20.38 | 708 | 1.220 | 0.964 |
| | Subst. | 76.7 | 20.38 | 708 | 1.214 | 0.946 |
| | " | 17.7 | 16.30 | 749 | 1.229 | 0.943 |
| 24 June " | " | 76.1 | 20.36 | 709 | 1.220 | 0.964 |
| | | 7.1 | 14.42 | 767 | 1.236 | 0.951 |

§ 4. *Agreement of the change of the dielectric constant of liquid oxygen below the boiling point with the formula of CLAUSIUS-MOSOTTI.* The determinations were performed at June 27 and July 12, 1922. In all of them the substitution method was employed. The object of the first was mainly to determine the dielectric constant at the boiling point and to investigate the influence of a large change in temperature, that of the second set in which greater accuracy of setting was obtained, to study the effect of a change in temperature in more detail. They are recorded in table II, that has been arranged in the same way as table I.

¹⁾ P. G. CATH and H. KAMERLINGH ONNES, these Proceedings 26, p. 490: Leiden Comm. No. 152a.

²⁾ E. MATHIAS, C. A. CROMMELIN and H. KAMERLINGH ONNES, these Proceedings 29, p. 935; Leiden Comm. No. 154b.

TABLE II.

| Liquid oxygen | | | | | |
|---------------|----------|---------------------------|-----------------------|-------|-------------------------------------|
| Date | Pressure | Temperature ¹⁾ | Density ²⁾ | K | $\frac{K-1}{K+2} \cdot \frac{1}{D}$ |
| 27 June 1922 | 77.2 cm | 90°.29 K | 1.145 | 1.463 | 0.1167 |
| | 15.2 | 77.12 | 1.210 | 1.493 | 0.1167 |
| | 13.2 | 76.19 | 1.214 | 1.496 | 0.1169 |
| 12 July „ | 76.0 | 90.11 | 1.146 | 1.464 | 0.1168 |
| | 41.0 | 84.61 | 1.174 | 1.478 | 0.1172 |
| | 25.0 | 80.70 | 1.193 | 1.487 | 0.1172 |
| | 9.6 | 74.16 | 1.223 | 1.504 | 0.1176 |
| | 5.4 | 70.75 | 1.239 | 1.504 | 0.1173 |

HASENÖHRL ³⁾ found at the boiling point 1.465.

The junior author held a National Research Fellowship (U. S. A.) while the above work was done.

¹⁾ P. G. CATH, these Proceedings 27, p. 553; Leiden Comm. No. 152*d*.

²⁾ E. MATHIAS and H. KAMERLINGH ONNES, these Proceedings 13, p. 939; Leiden Comm. No. 117.

³⁾ *l.c.*

Physics. — "*Further experiments with liquid helium. V. On the dielectric constant of liquid helium.*" By M. WOLKE and H. KAMERLINGH ONNES. Communication N°. 171*b* from the Physical Laboratory at Leiden.

(Communicated at the meeting of September 27, 1924).

We have determined the dielectric constant of liquid helium at the boiling point temperature under atmospheric pressure by means of high-frequency oscillations by a method that was elaborated by one of us in the Physical Institute of the Technical High School at Warsaw.

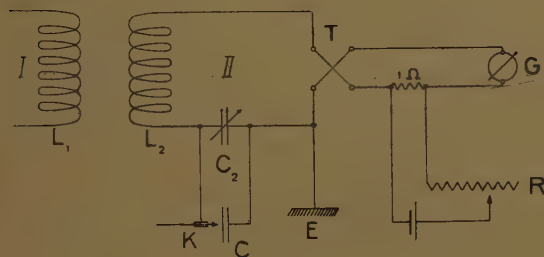


Fig. 1.

The arrangement of the experiment consists of two oscillatory circuits I and II (fig. 1), which are loosely coupled by means of the two self-inductions L_1 and L_2 .

In oscillatory circuit I undamped oscillations were excited in the well-known

way (shown diagrammatically in fig. 2) by the aid of the audion A . The true oscillatory circuit consists of the two self-inductions L and L_1 and the variable condenser C_1 . The filament of the audion A is earthed at E . We have chosen this method out of many others for the reason that experience has taught that it gives the greatest guarantee of the constancy of the oscillations¹⁾.

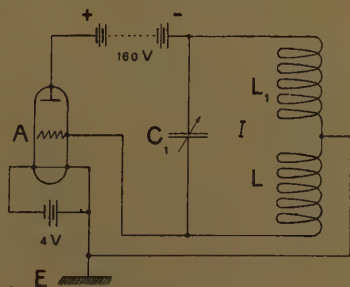


Fig. 2.

The oscillatory circuit II is the real measuring circuit; it consists of the self L_2 , a gauged variable condenser C_2 , and a vacuum thermo-element T , which is used here as detector. The measuring condenser C_2 is connected with the unknown capacity C ; by means of the adjustable contact K it can either be connected in parallel to the condenser C_2 , or separated from it. This capacity C is adapted for being filled with the substance under consideration,

¹⁾ Cf. G. HERWEG, Verh. d. D. Phys. Ges. 21 (1919) p. 572.

in our case with liquid helium, and placed in a cryostat, which is more fully described further down. E means the earthing. The galvanometer G serves to measure the e.m.f. of the thermo-element T ; the thermo-currents are compensated by regulating the resistance R in the compensation-connexion shown in fig. 1. The vacuum thermo-element used gave an e.m.f. of 7.5 mV at a current intensity of 10 mA in the heated resistance. The sensitiveness of the galvanometer G was $2 \cdot 10^{-9}$ A per mm scale division at 5.5 m scale distance. The measuring condenser C , could be accurately adjusted

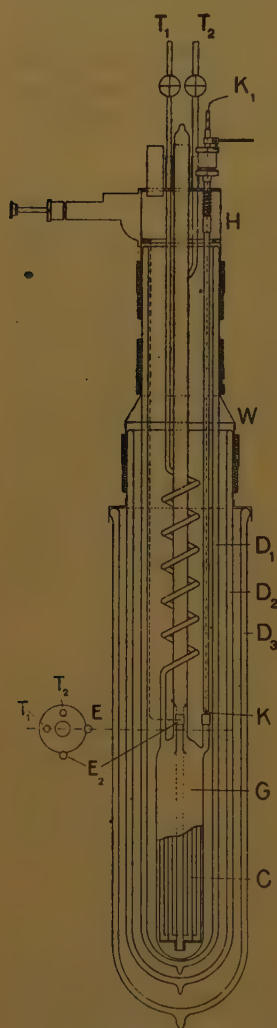


Fig. 3.

from the position of the observer by means of cords and turn tables. For the reading of its position serves a scale about 5 m long, which is placed in a circle of a radius of 2 m round the axis of the variable condenser. A small hollow mirror is fastened on the rotation axis of the measuring condenser, which throws a magnified image of a strongly illuminated slit 0.2 mm wide on the scale. 1 cm displacement of the image on the scale corresponds to a change of 0.518 cm in the capacity of the measuring condenser. A similar device was used by M. JEZEWSKI¹⁾; we have chosen here the objective reading for convenience sake.

The cryostat with the capacity C (fig. 1) is represented in fig. 3. It has been constructed according to our indications by Mr. G. J. FLIM, to whom we tender our best thanks. The condenser C ²⁾ consists of 5 brass cylinders, fastened concentrically into each other and alternately in conducting connection with each other; the outer cross-section of the largest cylinder is about 34 mm in diameter, the height of the cylinder about 100 mm., and the capacity of the whole condenser about 90 cm. This condenser is enclosed in a glass vessel G , silvered on the inside, which can be completely filled with the substance to be examined through the tubes T_1 and T_2 fused on to it. The glass vessel G is placed in two coaxial glass DEWAR

¹⁾ M. JEZEWSKI, Krakauer Akad. A. 60 (1923), p. 73.

²⁾ Devised by Dr. L. DANA.

vessels D_1 and D_2 , which are shut off on the outside by the caps H and W and are filled with liquid helium or hydrogen as may be required; the outer open DEWAR vessel D_3 is intended to receive liquid air. From the electrode E of the condenser C a wire insulated with respect to the silvered glass vessel G leads to a metal box, which is found immediately under the movable contact K . This contact K can either be brought in conducting connexion with the electrode E , or separated from it by turning the lever K_1 . The other electrode E_2 leads from the outer cylinder of the condenser C ; it is in conducting connexion with the inside coat of silver of the glass vessel G and with the two caps H and W ; and connected to the earthed side of the measuring condenser C_2 (fig. 1). This connexion with the earth forms a perfect protection for the condenser C against capacity influences from the outside.

The measurement of the capacity C (fig. 1) takes place in the following way. By means of the condenser C_1 (fig. 2) the oscillatory circuit I is adjusted at a constant wavelength, which had a value between 400 and 600 m. in these measurements. With open contact K (fig. 1) and a definite adjustment of the measuring condenser C_2 , the thermocurrent in the galvanometer circuit is compensated. Then the capacity C that is to be measured is connected in parallel with the measuring condenser C_2 by switching on the contact K , and the capacity of the condenser C_2 is diminished through rotation till the galvanometer G again reaches zero, through which the original intensity of the current in circuit II is restored. In this way the added capacity C is compensated by the corresponding diminution of the measuring condenser C_2 , so that the number of scale divisions, over which the image on the scale of the measuring condenser has been displaced on rotation of that condenser, indicates the required capacity in relative units.

The *dielectric constant* is calculated as ratio of the capacity of the condenser C , filled with the substance under investigation (here helium) to its capacity at the same temperature in vacuo. The vacuum is easily obtained in the glass vessel G (fig. 3) by means of a vacuum pump. As the connexion of the capacity C to the measuring condenser C_2 (fig. 1) is found in the box under K , attention has to be paid to the capacity of that box (connected to the electrode E (fig. 3)), which is included in the measurement of the capacity C , and can give rise to an error in the calculation of the dielectric constant. This error can be easily estimated; the capacity of the box is about: 0.45 cm., i.e. less than 0.6% of the capacity C ; as, however, the dielectric constant of the liquid helium according to

our measurements differs only by about 5 %, from that of the vacuum, the error in question becomes smaller than 0.03 %, i. e. less than one third in thousand.

The reading on the scale of the measuring condenser C , (fig. 1) is accurate only up to 1 mm., so that the accuracy of the measurements is about 0.1 % in the deflections observed by us, though the probable error is considerably smaller in all the series of measurements.

A similar method of measurement was also recently used by BRYAN¹⁾ for the determination of the dielectric constant.

Our method of measurement does not take into account the possible conduction current in condenser C . In order, therefore, to estimate the error that might arise in consequence of this, we have measured the resistance of the condenser C , filled with liquid helium, galvanometrically; we obtained about $2 \cdot 10^7$ Ohm. Availing ourselves of the known dimensions of condenser C , the specific resistance of liquid helium can be approximately determined; it is about $2.6 \cdot 10^{10}$ Ohm per cm^2 cross-section and cm length. It is seen from the above values that liquid helium is such a good insulator that a correction on account of the conduction current in the condenser is not required.

The sensitiveness of the measuring method used by us may be seen from this that with compensated thermocurrent a variation of the capacity of the measuring condenser of 1 cm on the measuring scale, i. e. of about 0.5 cm capacity, gave rise to a deviation of the galvanometer of about 6 cm on the galvanometer scale from the zero position.

To determine the dielectric constant of liquid helium at the temperature of its boiling point under atmospheric pressure the glass vessel G (fig. 3) and the DEWAR vessel D_1 were filled with liquid helium, and the capacity of the condenser C was measured in the liquid helium, the glass vessel being tightly closed; the helium evaporating slowly from the DEWAR vessel D_1 was collected in a gasometer. As a protection from heat from outside the DEWAR vessel D_2 was filled with liquid hydrogen, and the DEWAR vessel D_3 with liquid air.

The measurement took place with 6 adjustments of the condenser C_1 (fig. 2) in the generating circuit I, with 3 different wavelengths between 400 and 600 m. In each adjustment 6 readings were made, from which the arithmetical mean was taken. For each of these mean values the relative weight, as in inverse ratio to the sum of the squares of the deviations of the single readings from it, was

¹⁾ A. B. BRYAN, Phys. Rev. 22 (1923) p. 399.

calculated. In this way the following values (in cm of the measuring scale) were obtained for the capacity of the condenser C filled with liquid helium:

| | | | | | | |
|-----------|-------|-------|-------|-------|-------|-------|
| capacity: | 178.9 | 178.9 | 179.0 | 178.9 | 178.8 | 178.9 |
| weight: | 45.5 | 9.8 | 7.3 | 10.75 | 40.0 | 14.5. |

Making use of the relative weights, we obtain the following mean value for the capacity of the filled condenser from the above series of measurements:

$$178.9. \quad (1)$$

After completion of these measurements the helium was removed from the glass vessel G (fig. 3) by means of a vacuum-pump, and this well evacuated. Without making any change in the apparatus, we then measured the capacity of the condenser C in vacuo in the same way as described above. We obtained the following values:

| | | | | | | |
|-----------|-------|-------|-------|-------|-------|-------|
| capacity: | 170.5 | 170.7 | 170.8 | 171.1 | 170.2 | 170.7 |
| weight: | 20.0 | 66.7 | 2.58 | 8.2 | 4.98 | 12.8, |

from which follows as mean value:

$$170.7. \quad (2)$$

The required dielectric constant of liquid helium at the temperature of the boiling point under atmospheric pressure may be derived from the ratio of the two capacities (1) and (2):

$$K = \frac{178.9}{170.7} = 1.048.$$

The accuracy of the measurement being about 0.1%, we may put:

$$K = 1.048 \pm 0.001.$$

In these measurements the height of the barometer was 765 mm, corresponding to a boiling point of helium of 4.2° K ¹⁾.

In connexion with these measurements also a control measurement of the capacity of the evacuated condenser was performed at atmospheric temperature (about 25° C.). The following values were found:

| | | | | | | |
|-----------|-------|-------|--------|-------|-------|-------|
| capacity: | 171.6 | 171.7 | 171.65 | 171.7 | 171.7 | 171.8 |
| weight: | 12.7 | 8.34 | 8.47 | 9.8 | 2.08 | 7.15 |

with a mean value of:

$$171.7. \quad (3)$$

¹⁾ H. KAMERLINGH ONNES and S. WEBER, these Proceedings 18, p. 493; Leiden Comm. No. 147b.

The difference with the capacity at the boiling point of liquid helium may be explained by the change in dimensions caused by the temperature difference.¹⁾

¹⁾ This may be shown thus: the condenser is mounted on a glass rod, so a change of temperature is accompanied by two simultaneous changes of the capacity, one being caused by the change in dimensions of the condenser material (brass), the other being a consequence of the change in length of the glass rod. The latter may be shown easily to be equal to $(\alpha_{\text{brass}} - \alpha_{\text{glass}}) \cdot C \cdot \Delta T$, α being the mean coefficient of expansion in the temperature interval ΔT and C the capacity. As the change in capacity due to the brass on itself is equal to $\alpha_{\text{brass}} \cdot C \cdot \Delta T$, the total change ΔC is equal to $(2\alpha_{\text{brass}} - \alpha_{\text{glass}}) \cdot C \cdot \Delta T$.

Applying GRÜNEISEN'S rule concerning the proportionality between specific heat and coefficient of expansion of copper, one finds

$$\alpha_{\text{Cu}, 0^{\circ}-300^{\circ} \text{ K.}} = 12.6 \cdot 10^{-6}$$

and

$$\alpha_{\text{Cu}, 80^{\circ}-290^{\circ} \text{ K.}} = 15.9 \cdot 10^{-6}.$$

Now HENNING gives (LANDOLT-BÖRNSTEIN, Tables II (1923), p. 1219)

$$\alpha_{\text{brass}, 82^{\circ}-289^{\circ} \text{ K.}} = 16.6 \cdot 10^{-6}.$$

So we put

$$\alpha_{\text{brass}, 0^{\circ}-300^{\circ} \text{ K.}} = 12.6 \cdot 16.6/15.9 \cdot 10^{-6} = 13.1 \cdot 10^{-6}.$$

As to glass, from as yet unpublished determinations of KAMERLINGH ONNES and VAN AGT on the thermal expansion of Jena 16^{III} we infer

$$\alpha_{\text{glass}, 0^{\circ}-300^{\circ} \text{ K.}} = 4.7 \cdot 10^{-6}.$$

(For our case this value will probably be too low, as the glass actually used was Thüringer glass, which has a larger coefficient of expansion, at least at atmospheric temperature).

So we get for a temperature difference of 294 degrees:

$$\Delta C = (2 \cdot 13.1 - 4.7) \cdot 10^{-6} \cdot 170.7 \cdot 294 = 1.08$$

the difference observed being 1.0. [Note added in the translation].

Physics. — "*On the dielectric constants of liquid and solid hydrogen*".

By M. WOLFKE and H. KAMERLINGH ONNES. Communication
Nº. 171c from the Physical Laboratory at Leiden.

(Communicated at the meeting of September 27, 1924.)

Preliminary experiments on the dielectric constant of liquid hydrogen were already earlier undertaken by G. BREIT and H. KAMERLINGH ONNES¹⁾. We have now carried out these measurements with greater accuracy, and also determined the dielectric constant of solid hydrogen.

These measurements were made by the same method and with the same apparatus as was employed in the determination of the dielectric constant of liquid helium²⁾; we may, therefore, refer to this work for all particulars.

The dielectric constant was calculated as ratio of the capacity of the condenser filled with liquid, resp. solid hydrogen to its capacity at the same temperature in vacuo. The accuracy of the measurement is about 0.1 %.

In order to estimate the possible error which might arise through the conduction current in the filled condenser, we have approximately determined the specific resistance of the liquid and the solid hydrogen galvanometrically. We obtained the following results: 1.3×10^9 Ohm for liquid hydrogen and more than 10^{11} Ohm for solid hydrogen calculated per cm² cross section and cm length. Hence both in liquid and in solid condition hydrogen is such a good insulator, that there is no call for a correction on account of the conduction in the condenser.

The measurements were made with three different wavelengths, between 400 and 600 m. wavelength. For every wavelength six measurements were made, and the arithmetical mean was taken from these values; the final value was calculated from these means by the method of least squares.

In the measurements at the temperature of the boiling point under

¹⁾ These Proceedings 27, p. 617, Leiden Comm. No. 171a.

²⁾ M. WOLFKE and H. KAMERLINGH ONNES, these Proceedings 27, p. 621; Leiden Comm. No. 171b.

atmospheric pressure the glass vessel¹⁾, containing the condenser was filled with liquid hydrogen, and tightly closed. This vessel was placed in a DEWAR vessel, also filled with liquid hydrogen, which could evaporate freely into the air. A DEWAR vessel with liquid air served as protection against the heat from outside. For the capacity of the condenser filled with hydrogen we obtained in this way the following values in cm of the measuring scale:

| | | | | | | |
|-----------|-------|-------|-------|-------|-------|-------|
| capacity: | 210.2 | 210.4 | 210.6 | 210.3 | 210.5 | 210.1 |
| weight: | 5.56 | 8.85 | 1.37 | 45.5 | 24.4 | 8.92. |

Taking the given weights into account, we get as mean value:

$$210.3. \quad (1)$$

After these measurements had been completed, the hydrogen was removed from the condenser vessel by means of a vacuum pump, and then the vessel was evacuated. In the same way as before, the capacity of the evacuated condenser was determined; we obtained the following values:

| | | | | | | |
|-----------|-------|-------|-------|-------|-------|-------|
| capacity: | 171.7 | 171.9 | 171.8 | 172.1 | 171.5 | 172.0 |
| weight: | 33.3 | 0.99 | 2.38 | 1.91 | 9.9 | 8.47, |

with a corresponding mean value of:

$$171.7. ^2) \quad (2)$$

The ratio of the two capacities (1) and (2) gives the required value of the dielectric constant of liquid hydrogen:

$$K = \frac{210.3}{171.7} = 1.225. \quad (3)$$

A second series of values measured in the same way gave the following value for the dielectric constant of the liquid hydrogen

$$K = 1.224. \quad (4)$$

These two values (3) and (4) differ less from each other than 0.08 %. We consider the first value (3) as more accurate, as in the second measurement the determination of the capacity of the evacuated condenser did not immediately follow the measurement of the capacity of the filled condenser, but was made later. The dielectric constant

¹⁾ M. WOLFKE and H. KAMERLINGH ONNES, loc. cit. cf. fig. 3.

²⁾ The coincidence with the value found for the capacity of the evacuated condenser at atmosphere temperature, given in the preceding communication, is only accidental; cf. also the following footnote.

of liquid hydrogen at the boilingpoint under atmospheric pressure has thus the following value:

$$K = 1.225 \pm 0.001.$$

Besides those determinations we have carried out experiments on the dependence of the dielectric constant of liquid and solid hydrogen on temperature.

For this purpose the DEWAR vessel containing the hydrogen bath, in which the glass vessel with the condenser was placed, was brought in connexion with a vacuum pump. Through sucking off of the evaporating hydrogen the vapour pressure could be reduced, and thus the temperature of the hydrogen bath could be lowered. An arrangement regulating the velocity of the sucking off of the hydrogen vapour enabled us to keep the pressure and the temperature constant throughout a measurement.

On account of the slight expansion of metals at low temperatures the capacity of the evacuated condenser was assumed to be constant, and put at the value 170.7 like the value found before.¹⁾

The temperatures corresponding to the measured pressures of the hydrogen vapour in equilibrium with the liquid phase have been taken from the work of P. G. CATH and H. KAMERLINGH ONNES²⁾.

The temperatures corresponding to the vapour pressures of the solid phase, were calculated from J. E. VERSCHAFFELT's equation³⁾.

The measurements of the dielectric constant in these experiments do not claim the same degree of accuracy as the earlier ones, especially those of solid hydrogen at low temperatures are not very accurate. The measurements had to be made very quickly, and therefore only two readings were made for each value, and there is no guarantee that temperature equilibrium between the hydrogen bath and the interior of the condenser had been sufficiently reached.

Table I gives a survey of the results obtained.

It follows from the above values that *the dielectric constant of liquid hydrogen increases with falling temperature to within the neigh-*

¹⁾ Viz. at the boiling point of helium, cf. the preceding communication. The determinations at the latter temperature have been preceded by those at the boiling point of liquid hydrogen and have been followed by those at the lower hydrogen temperatures. Between the first and the second series of measurements the relative value of the capacity has changed a small amount (from 171.7 to 170.7), the apparatus having been modified somewhat in the mean time.

²⁾ P. G. CATH and H. KAMERLINGH ONNES, these Proceedings 20, p. 1155; Leiden Comm. No. 152a.

³⁾ Fourth Intern. Congres of Refrig. London 1924, Reports and Comm. First Intern. Comm. Intern. Inst. of Refr. No. 2, p. 23; Arch. Néerl. d. Sc. ex. et nat. (III A) 8 (1924).

TABLE I.

| Bath | Pressure of the bath | T . | K |
|-----------------|-------------------------|---------|-------|
| liquid hydrogen | 755 mm. Hg | 20.33 K | 1.225 |
| " " | 357 | 18.05 | 1.234 |
| " " | 80 | 14.64 | 1.241 |
| solid " | 58 | 14.0 | 1.248 |
| " " | 42 | 13.5 | 1.224 |
| " " | 35 | 13.3 | 1.212 |
| " " | 32 | 13.2 | 1.211 |

bourhood of the meltingpoint, where it reaches its highest value, after which it decreases in solid hydrogen on further decrease of temperature.

This behaviour is often shown by the dielectric constant on transition of a substance from the liquid phase to the solid phase¹⁾.

We will also use the values of the dielectric constant of hydrogen to test the CLAUSIUS-MOSOTTI formula. Only the first three values of the table, which refer to the liquid state of hydrogen, can be used for this purpose. We derive the corresponding densities of the liquid hydrogen by interpolation from the results of E. MATHIAS, C. A. CROMMELIN and H. KAMERLINGH ONNES²⁾, and we then calculate the constants of the CLAUSIUS-MOSOTTI formula:

$$\frac{K-1}{K+2} \cdot \frac{1}{D} = \text{const.}$$

The calculated values are recorded in the following table.

TABLE II.

| T | D | K | $\frac{K-1}{K+2} \cdot \frac{1}{D}$ |
|---------|--------|-------|-------------------------------------|
| 20.33 K | 0.0709 | 1.225 | 0.984 |
| 18.05 | 0.0733 | 1.234 | 0.987 |
| 14.64 | 0.0765 | 1.241 | 0.973 |

We see from this that the CLAUSIUS-MOSOTTI formula is satisfied to about 1% in liquid hydrogen.

¹⁾ Cf. H. ISNARDI, Zs. f. Phys. 9 (1922) p. 153.

²⁾ These Proceedings 29, p. 935; Leiden Comm. No. 154b.

Mathematics. — "*Intuitionistische Ergänzung des Fundamentalsatzes der Algebra*". By Prof. L. E. J. BROUWER.

(Communicated at the meeting of May 31, 1924).

§ 1.

Unter einer *r*-normalen Gleichung *n*-ten Grades verstehen wir eine algebraische Gleichung *n*-ten Grades $f(x) \equiv a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$, deren Koeffizient a_r von 0 positiv-verschieden ist. Eine 0-normale Gleichung *n*-ten Grades werden wir kurz *normal* nennen. Auf Grund des (beschränkten) Fundamentalsatzes der Algebra¹⁾ ist dann bekannt, dass das linke Glied einer normalen Gleichung *n*-ten Grades in ein Produkt $a_0 (x - \lambda_1) (x - \lambda_2) \dots (x - \lambda_n)$, wo $\lambda_1, \lambda_2, \dots, \lambda_n$ Zahlen vorstellen, entwickelt werden kann. Unter Verwendung dieser Eigenschaft werden wir jetzt von einer *r*-normalen Gleichung ($r \leq n-1$)

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0$$

eine Wurzel α_1 bestimmen. Wir werden diese Wurzel erzeugen als Limes einer positiv-konvergenten Reihe q_1, q_2, q_3, \dots , in welcher für jedes *v*

$$q_{v+1} - q_v \leq 2^{-v},$$

während überdies sogar, *ausgenommen für höchstens n Werte von v*, $q_{v+1} - q_v = 0$.

Dabei wird jedesmal q_{m+1} folgendermassen aus q_m hergeleitet: Wenn wir für eine gewisse natürliche Zahl $m \geq 1$ über q_m verfügen als Wurzel der normalen Gleichung

$$f_m(x) \equiv a_{\sigma_m} x^{n-\sigma_m} + a_{\sigma_m+1} x^{n-\sigma_m-1} + \dots + a_n = 0 \quad (\sigma_m \leq n-1),$$

deren linkes Glied ein Endsegment von $f(x)$ darstellt, während für $0 \leq \tau < \sigma_m$ derartige positive Grössen ${}_m g_\tau$ bekannt sind, dass $|a_\tau| < {}_m g_\tau$ ($0 \leq \tau < \sigma_m$) und jede *normale* Gleichung

$$b_0 x^n + \dots + b_{\sigma_m-1} x^{n-\sigma_m+1} + f_m(x) = 0,$$

in welcher $|b_\tau| < {}_m g_\tau$ ($0 \leq \tau < \sigma_m$), eine um höchstens 2^{-m} von q_m verschiedene Wurzel besitzt, so bestimmen wir q_{m+1} als Wurzel einer normalen Gleichung

$$f_{m+1}(x) \equiv a_{\sigma_{m+1}} x^{n-\sigma_{m+1}} + \dots + a_n = 0 \quad (\sigma_{m+1} \leq \sigma_m),$$

¹⁾ Vgl. die in diesem Jahre erschienenen Beweise von H. WEYL, Math. Zeitschr. 20, S. 142—146, und von L. E. J. BROUWER und B. DE LOOR, diese Proceedings 27, S. 186—188.

wobei für $0 \leq \tau < \sigma_{m+1}$ derartige positive Grössen $_{m+1}g_\tau$ bekannt sind, dass $|a_\tau| < _{m+1}g_\tau$ ($0 \leq \tau < \sigma_{m+1}$) und jede *normale* Gleichung

$$b_0 x^n + \dots b_{\sigma_{m+1}-1} x^{n-\sigma_{m+1}+1} + f_{m+1}(x) = 0,$$

in welcher $|b_\tau| < _{m+1}g_\tau$ ($0 \leq \tau < \sigma_{m+1}$) eine um höchstens 2^{-m-1} von q_{m+1} verschiedene Wurzel besitzt.

Diese Bestimmung geschieht wie folgt: Wir schlagen in der x -Ebene um q_m mit einem Radius $\leq 2^{-m-1}$ einen Kreis, der von allen Wurzeln von $f_m(x) = 0$ einen von 0 positiv-verschiedenen Abstand besitzt. Sei k_m das vom absoluten Werte von f_m auf diesem Kreise angenommene Minimum und ${}_m h_\tau$ für $0 \leq \tau < \sigma_m$ eine solche positive Grösse, dass auf dem genannten Kreise

$$|b_0 x^n + \dots b_{\sigma_m-1} x^{n-\sigma_m+1}| < k_m$$

für $|b_\tau| < {}_m h_\tau$ ($0 \leq \tau < \sigma_m$). Alsdann besitzt jede *normale* Gleichung

$$b_0 x^n + \dots b_{\sigma_m-1} x^{n-\sigma_m+1} + f_m(x) = 0,$$

für welche $|b_\tau| < {}_m h_\tau$ ($0 \leq \tau < \sigma_m$), innerhalb des genannten Kreises eine Wurzel.

Jetzt können wir *entweder* die Ungleichungen $|a_\tau| < {}_m h_\tau$ ($0 \leq \tau < \sigma_m$) herleiten, in welchem Falle wir, um unser Ziel zu erreichen, nur $q_{m+1} = q_m$ und $f_{m+1}(x) \equiv f_m(x)$ zu setzen brauchen, *oder* eine *normale* Gleichung

$$f'_m(x) \equiv a_{\sigma'_m} x^{n-\sigma'_m} + \dots a_n = 0 \quad (\sigma'_m < \sigma_m),$$

deren linkes Glied wiederum ein Endsegment von $f(x)$ darstellt, bilden, und von derselben eine um höchstens 2^{-m} von q_m verschiedene Wurzel q'_m angeben.

Im letzteren Falle verfahren wir mit f'_m und q'_m genau so, wie vorhin mit f_m und q_m , schlagen also in der x -Ebene um q'_m mit einem Radius $\leq 2^{-m-1}$ einen Kreis, der von allen Wurzeln von $f'_m(x) = 0$ einen von 0 positiv-verschiedenen Abstand besitzt. Sei k'_m das vom absoluten Werte von f'_m auf diesem Kreise angenommene Minimum und ${}_m h'_\tau$ für $0 \leq \tau < \sigma'_m$ eine solche positive Grösse, dass auf dem genannten Kreise

$$|b_0 x^n + \dots b_{\sigma'_m-1} x^{n-\sigma'_m+1}| < k'_m$$

für $|b_\tau| < {}_m h'_\tau$ ($0 \leq \tau < \sigma'_m$). Alsdann besitzt jede *normale* Gleichung

$$b_0 x^n + \dots b_{\sigma'_m-1} x^{n-\sigma'_m+1} + f'_m(x) = 0,$$

für welche $|b_\tau| < {}_m h'_\tau$ ($0 \leq \tau < \sigma'_m$), innerhalb des genannten Kreises eine Wurzel.

Jetzt können wir wieder *entweder* die Ungleichungen $|\alpha_\tau| < {}_m h'_\tau$ ($0 \leq \tau < \sigma'_m$) herleiten, in welchem Falle wir, um unser Ziel zu erreichen, nur $\varrho_{m+1} = \varrho'_m$ und $f_{m+1}(x) \equiv f'_m(x)$ zu setzen brauchen, *oder* eine normale Gleichung

$$f''_m(x) \equiv a_{\sigma''_m} x^{n-\sigma''_m} + \dots a_n = 0 \quad (\sigma''_m < \sigma'_m),$$

deren linkes Glied wiederum ein Endsegment von $f(x)$ darstellt, bilden, und von derselben eine um höchstens 2^{-m} von ϱ_m verschiedene Wurzel ϱ''_m angeben.

Im letzteren Falle verfahren wir mit f''_m und ϱ''_m genau so, wie vorhin mit f_m und ϱ_m und mit f'_m und ϱ'_m , und finden dann, dass wir *entweder*, um unser Ziel zu erreichen, nur $\varrho_{m+1} = \varrho''_m$ und $f_{m+1}(x) \equiv f''_m(x)$ zu setzen brauchen, *oder* eine normale Gleichung

$$f'''_m(x) \equiv a_{\sigma'''_m} x^{n-\sigma'''_m} + \dots a_n = 0 \quad (\sigma'''_m < \sigma''_m)$$

bilden und von derselben eine um höchstens 2^{-m} von ϱ_m verschiedene Wurzel ϱ'''_m angeben können.

Indem wir in dieser Weise fortfahren, gelangen wir für ein gewisses $\mu_m \leq n-1$ zu einer solchen normalen Gleichung $f_m^{(\mu_m)}(x)$ mit einer solchen um höchstens 2^{-m} von ϱ_m verschiedenen Wurzel $\varrho_m^{(\mu_m)}$, dass wir, um unser Ziel zu erreichen, nur $\varrho_{m+1} = \varrho_m^{(\mu_m)}$ und $f_{m+1}(x) \equiv f_m^{(\mu_m)}(x)$ zu setzen brauchen.

Zur Vervollständigung der Berechnungsmethode für die Wurzel α_1 von $f(x) = 0$ braucht nunmehr nur noch ein ϱ_1 und $f_1(x)$ lieferndes Verfahren angegeben zu werden. Dazu gehen wir aus von einer auf jedem Fall bestehenden normalen Gleichung

$$f_0(x) \equiv a_{\sigma_0} x^{n-\sigma_0} + a_{\sigma_0+1} x^{n-\sigma_0-1} + \dots a_n = 0 \quad (\sigma_0 \leq n-1),$$

deren linkes Glied ein Endsegment von $f(x)$ darstellt, und von einer beliebigen Wurzel ϱ_0 von $f_0(x)$. Aus ϱ_0 und $f_0(x)$ leiten wir dann $\varrho_1 = \varrho_0^{(\mu_0)}$ und $f_1(x) \equiv f_0^{(\mu_0)}(x)$ in analoger Weise her, wie wir oben, aus ϱ_m und $f_m(x)$, $\varrho_{m+1} = \varrho_m^{(\mu_m)}$ und $f_{m+1}(x) \equiv f_m^{(\mu_m)}(x)$ hergeleitet haben.

§ 2.

Auf Grund des obigen stellen wir uns zum Ziel, eine r -normale ($r \leq n$) Gleichung n -ten Grades

$$f(x) \equiv \alpha_0 x^n + \alpha_1 x^{n-1} + \dots \alpha_n = 0$$

in n Faktoren ersten Grades zu zerlegen. Hierbei dürfen wir annehmen $r \leq n-1$, weil für $r = n$ die Zerlegung schon auf Grund des (beschränkten) Fundamentalsatzes der Algebra, und zwar in der Form $f(x) \equiv (1-\beta_1 x)(1-\beta_2 x) \dots (1-\beta_n x) a_n$, bewerkstelligt werden kann. Wenn aber $r \leq n-1$, so können wir nach § 1 eine Wurzel α_1 von $f(x) = 0$ berechnen, und weiter ein solches a_s ($0 \leq s \leq r$) angeben, dass

$$\frac{|a_s|}{\alpha_1^s} > \sum_{v=0}^{s-1} |a_v| < |a_s|,$$

wobei $\sum_{v=0}^{s-1} |a_v| = 0$ für $s=0$ gerechnet wird. In der mittels des Divisionsalgorithmus gefundenen Zerlegung $f'(x) \equiv (x-\alpha_1)(c_0 x^{n-1} + c_1 x^{n-2} + \dots c_{n-1})$ stellt nun $c_0 x^{n-1} + \dots c_{n-1} = 0$ eine s -normale ($s \leq r$) Gleichung dar. Für die letztere Gleichung können wir *entweder* (für $s = n-1$) auf Grund des (beschränkten) Fundamentalsatzes der Algebra eine Zerlegung in $n-1$ Linearfaktoren, *oder* (für $s \leq n-2$) eine Wurzel α_s bestimmen, worauf wir im letzteren Falle mit ihr genau so weiter verfahren können, wie vorhin mit $f(x) = 0$ geschehen ist.

Indem wir in dieser Weise fortfahren, gelangen wir schliesslich zu einer Zerlegung in Faktoren folgender Form:

$$f(x) \equiv (x-\alpha_1)(x-\alpha_2) \dots (x-\alpha_p)(1-\beta_{p+1}x) \dots (1-\beta_n x) c, \quad (1)$$

wo c positiv-verschieden von 0 und $p \geq n-r$ ist. Daneben gibt es, wie sich in derselben Weise herausstellt, eine Zerlegung gleicher Gestalt, für welche aber $p \leq n-r$ ist. Mithin kann, wie eine kurze Ueberlegung zeigt, die Zerlegung in der Form (1) auch so bewerkstelligt werden, dass $p = n-r$ ist.

Mathematics. — “*Bemerkungen zum natürlichen Dimensionsbegriff.*”

By Prof. L. E. J. BROUWER.

(Communicated at the meeting of June 28, 1924).

§ 1.

Wir beschränken uns im folgenden auf *kondensierte Spezies*, d.h. abgeschlossene kompakte metrische Spezies oder Vereinigungen von divergenten Fundamentalreihen abgeschlossener kompakter metrischer Spezies, und betrachten für diese Spezies den vor einiger Zeit von MENGER¹⁾ und URYSOHN²⁾ eingeführten Dimensionsbegriff, den wir den *MU-Dimensionsbegriff* nennen werden. Die Definition dieses Begriffes hat für kondensierte Spezies folgende Form:

“Eine Spezies π hat eine *MU-Dimension* $\leq n$, wenn ein beliebiger Punkt m von π für beliebiges positives ε eine Umgebung $U(m) < \varepsilon$ besitzt, deren Grenze eine *MU-Dimension* $\leq n-1$ besitzt. Eine Spezies π hat die *MU-Dimension* 0, wenn sie kein Kontinuum als Teil enthält”.

Mittels vollständiger Induktion in bezug auf n zeigt man leicht, dass die Vereinigung einer endlichen Anzahl oder einer divergenten Fundamentalreihe von Spezies der *MU-Dimension* $\leq n$ wiederum eine Spezies der *MU-Dimension* $\leq n$ darstellt.

Wir betrachten nun in einer Spezies π der *MU-Dimension* $\leq n$ zwei innerhalb π abgeschlossene Teilspezies ϱ und ϱ' von π , welche keine gemeinsamen Punkte besitzen. Dann folgt aus der Verallgemeinerung des HEINE-BORELSchen Theorems, dass es eine Umgebung $U(\varrho)$ von ϱ gibt, welche eine Grenze π_1 der *MU-Dimension* $\leq n-1$ besitzt und von der ϱ' weder einen Punkt noch einen Grenzpunkt enthält. M. a. W. ϱ und ϱ' können in π getrennt werden³⁾ durch eine innerhalb π abgeschlossene Teilspezies π_1 von π der *MU-Dimension* $\leq n-1$.

Wenn wir also einer Spezies π eine *N-Dimension* $\leq n$ zusprechen, wenn sie nach der Terminologie des in³⁾ zitierten Aufsatzes einen *allgemeinen Dimensionsgrad* $\leq n$ besitzt, so haben wir:

¹⁾ Monatshefte für Mathematik und Physik, Bd. 33, S. 157—160 (nach in 1921 bei der Wiener Akademie hinterlegten Definitionen und Resultaten).

²⁾ Comptes Rendus, t. 175 (1922), S. 440.

³⁾ Diese Proceedings 26, S. 796. Hinsichtlich einer hier in bezug auf das 1913 im Journ. f. Math. 142 erschienene Original vorgenommenen Berichtigung vgl. näher § 2, sowie meine in Bd. 21 der Mathem. Zeitschr. erscheinende Notes „zum natürlichen Dimensionsbegriff”.

Eine Spezies der MU-Dimension $\leq n$ besitzt auch eine N-Dimension $\leq n$.

Weil mittels vollständiger Induktion in bezug auf n auch die Umkehrung dieses Satzes leicht bewiesen werden kann, so ergeben für kondensierte Spezies die Definitionen der MU-Dimension und der N-Dimension den gleichen Dimensionsbegriff⁴⁾. Die Bezeichnung als „natürlicher“ Dimensionsbegriff scheint mir für diesen Begriff in erster Linie auf Grund der ihm anhaftenden Qualität, welche durch den N-Dimensionsbegriff zum Ausdruck kommt, berechtigt zu sein.

§ 2.

Dem natürlichen Dimensionsbegriff liegt folgende, der Anschauung entnommene „natürliche Trennungsdefinition“ zu grunde:

a) Wenn π_1 , ρ und ρ' innerhalb π abgeschlossene Teilspezies von π ohne gemeinsame Punkte sind, so heissen ρ und ρ' in π durch π_1 getrennt, wenn jede zusammenhängende Teilspezies von π , welche sowohl mit ρ wie mit ρ' Punkte gemeinsam hat, auch von π_1 mindestens einen Punkt enthält⁵⁾.

In Journ. f. Math. 142, S. 147 hatte sich in diese Definition ausser jeder Beziehung zum übrigen Inhalte der Abhandlung das Wort „abgeschlossene“ eingeschlichen⁶⁾, wodurch die Trennungsdefinition folgende Form erhielt:

b) Wenn π_1 , ρ und ρ' innerhalb π abgeschlossene Teilspezies von

⁴⁾ Diese Äquivalenz wurde von Herrn URYSOHN schon im Herbst 1923 erwähnt.

⁵⁾ Im in Fussnote ³⁾ zitierten Aufsatz befindet sich diese Trennungsdefinition in der folgenden, mit der obigen äquivalenten, weniger anschaulichen, aber üblicheren und sich an die Beweisführung enger anschliessenden Form:

c) Wenn π_1 , ρ und ρ' innerhalb π abgeschlossene Teilspezies von π ohne gemeinsame Punkte sind, so heissen ρ und ρ' in π durch π_1 getrennt, wenn π_1 in π eine ρ enthaltende, aber ρ' nicht enthaltende Gebietsmenge bestimmt.

⁶⁾ Die Veröffentlichung der (übrigens für den Leser auf Grund des Zusammenhanges auf der Hand liegenden) Berichtigung dieses Schreibfehlers scheint vor elf Jahren dadurch unterblieben zu sein, dass ich im Anschluss an die damals in Aussicht gestellten (a. a. O. S. 151 in einer Fussnote beiläufig berührten) näheren Ausführungen LEBESGUES auf die Dimensionstheorie zurückzukommen und dann gleichzeitig die betreffende Korrektur auszuführen beabsichtigte, dass aber, als diese Ausführungen LEBESGUES Jahr aus Jahr ein auf sich warten liessen und meine Aufmerksamkeit von anderen Gegenständen in Anspruch genommen wurde, die Angelegenheit in Vergessenheit geraten ist. Erst anlässlich eines im vorigen Jahre auf der Jahresversammlung der Deutschen Mathematiker-Vereinigung in Marburg von Herrn URYSOHN gehaltenen Vortrags, in welchem auf die a. a. O. zu hebende Unstimmigkeit hingewiesen wurde, hat neuerdings im in ³⁾ zitierten Aufsatz, sowie in Journ. f. Math. 153 die Richtigstellung stattgefunden.

π ohne gemeinsame Punkte sind, so heissen q und q' in π durch π_1 getrennt, wenn jede zusammenhängende, abgeschlossene Teilspezies von π , welche sowohl mit q wie mit q' Punkte gemeinsam hat, auch von π_1 mindestens einen Punkt enthält.

Falls man, was ebenfalls möglich ist, die abstrakte Dimensionstheorie auf die letztere Definition gründen will, so ist vom in ⁷⁾ zitierten Aufsatz der zwischen S. 798 Z. 4 v. u. und S. 799 Z. 14 v. u. enthaltene Passus wie folgt zu ändern ⁷⁾:

“... bezeichnen wir mit τ das Ausgangselement $E_1 E_2 E_3 \dots E_{n+1}$, für jedes v mit τ_v den zu τ gehörigen Teil von π_v , konstruieren in τ eine simpliziale Zerlegung ζ von der Dichte ε , setzen $\gamma_0 = \tau$, und wählen für jedes v , nachdem $\gamma_0, \gamma_1, \gamma_2, \dots, \gamma_v$ bestimmt sind, für γ_{v+1} eine solche Vereinigung von zu γ_v gehörigen Grundsimplex von ζ , dass *erstens* die zu den Elementseiten $E_1 E_2 \dots E_{v+1}$ und $E_1 \dots E_v E_{v+2} \dots E_{n+1}$ gehörigen Teile von γ_v in γ_v durch γ_{v+1} getrennt sind, *zweitens* falls τ_{v+1} existiert, γ_{v+1} für passend gewähltes positives ε_{v+1} besteht aus denjenigen zu γ_v gehörigen Grundsimplex von ζ , welche einen Abstand $\leq \varepsilon_{v+1}$ von τ_{v+1} besitzen. *Als dann können wir $\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots$ alle mit ε gegen Null konvergieren lassen.*

Nehmen wir nämlich einen Augenblick an, dass wir $\varepsilon, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_v$ zusammen gegen Null konvergieren lassen können, während für $\varepsilon, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_v, \varepsilon_{v+1}$ dasselbe unmöglich ist, so lässt sich eine positive Grösse a angeben mit der Eigenschaft, dass für jedes positive η in τ_v in einer Entfernung $> a$ von τ_{v+1} eine η -Kette existiert, von deren Endpunkten das eine in einer Entfernung $< \eta$ von $E_1 E_2 \dots E_v E_{v+1}$ und das andere in einer Entfernung $< \eta$ von $E_1 \dots E_v E_{v+2} \dots E_{n+1}$ gelegen ist. Dann aber müsste in τ_v in einer Entfernung $\geq a$ von τ_{v+1} ein sowohl $E_1 \dots E_v E_{v+1}$ wie $E_1 \dots E_v E_{v+2} \dots E_{n+1}$ treffendes abgeschlossenes Kontinuum existieren, was ungereimt ist.

⁷⁾ Dass in dieser Weise die Tragweite des Textes nicht nur formal, sondern auch materiell beeinflusst wird, erläutert folgendes von Herrn URYSOHN herrührendes Beispiel: Es sei π die Euklidische Ebene; $E_1 = (0, -1)$, $E_2 = (3, 2)$, $E_3 = (-3, 2)$; π_1 die Vereinigung der folgenden sechs Punktmengen: 1. $x = 0$, $0 \leq y \leq 1$; 2. $y = 0$, $-1 \leq x \leq 0$; 3. $x = -1$, $-2 \leq y \leq 0$; 4. $y = -2$, $-1 \leq x \leq 1$; 5. $x = 1$, $-2 \leq y \leq 0$; 6. $y = \sin^3 \frac{\pi}{x}$, $0 < x \leq 1$; π_2 die der Trennungsdefinition

b), nicht aber der Trennungsdefinition a) genügende (vgl. in diesem Zusammenhang HAUSDORFF, Grundzüge der Mengenlehre, S. 458) Vereinigung der beiden Punkte $(0, -2)$ und $(0, 1)$. Die durch die alte Wendung des Textes a. a. O. S. 798—799 geforderte, in τ_1 von π_2 bestimmte, an $E_1 E_2$ grenzende, an $E_1 E_3$ jedoch nicht grenzende Gebietsmenge g_1 (durch welche das im Hilfssatz auftretende, aus Grundsimplex von σ_1 gebildete Fragment γ_1 , das alle an $E_1 E_2$, aber kein an $E_1 E_3$ grenzendes Grundsimplex von σ_1 enthält, bedingt ist) *existiert in diesem Falle nicht.*

Weil wir also $\varepsilon, \varepsilon_1, \varepsilon_2, \varepsilon_3, \dots$ in der Tat alle zusammen gegen Null konvergieren lassen können, so können wir, falls τ_{v_1} verschwindet, dafür sorgen, dass auch γ_{v_1} verschwindet. Der Dimensionssatz ist hiermit zurückgeführt auf den Nachweis der Eigenschaft, dass γ_{v_1} für $v_1 \leq n$ unmöglich verschwinden kann, und hiermit auf den Beweis von folgendem

Hilfssatz. Es sei σ ein simplizial zerlegtes n -dimensionales Element mit den Eckpunkten $E_1, E_2, \dots, E_{n+1}; \dots$."

Mathematics. — “*Ueber die Dimension von Punktmengen*”. By
Dr. K. MENDER. (Communicated by Prof. L. E. J. BROUWER).

(Communicated at the meeting of September 27, 1924).

Im Gegensatz zu den meisten bisherigen Definitionen der Dimension, welche sich auf den Abbildungsbegriff stützen, gehen wir von einer Definition aus, welche die Dimension der Mengen gleichsam an sich, nicht in Bezug auf einen willkürlich vorgegebenen Abbildungsbereich, und in Übereinstimmung mit unserer Anschauung festlegt. Wir betten die auftretenden Mengen in einen metrischen Raum, doch bleiben zahlreiche Ergebnisse auch bei Zugrundelegung allgemeiner topologischer Räume gültig.

Wir nennen eine Menge M *n-dimensional*, wenn n die kleinste Zahl ist von folgender Eigenschaft: Zu jedem Punkt von M existiert eine auf den betreffenden Punkt sich zusammenziehende Folge von Umgebungen (d. h. von offenen Mengen, die den Punkt enthalten), mit deren Begrenzungen M höchstens $(n-1)$ -dimensionale Durchschnitte hat. (-1) -dimensional und höchstens (-1) -dimensional ist die leere Menge und nur diese. Eine Menge M heisst *k-dimensional im Punkt m*, wenn k die kleinste Zahl ist, so dass eine auf m sich zusammenziehende Folge von Umgebungen existiert, mit deren Begrenzungen M höchstens $(k-1)$ -dimensionale Durchschnitte hat ¹⁾.

Zunächst ergibt sich die *Invarianz der Dimensionszahl* gegenüber topologischen (d. h. beiderseits stetigen und eindeutigen) Abbildungen. Ist $\{U_n(m)\}$ ($n = 1, 2, \dots$) eine auf den Punkt m sich zusammenziehende Folge von Umgebungen, B_n die Begrenzung von $U_n(m)$, und wird die Menge M topologisch abgebildet auf $M' = A(M)$,

¹⁾ Nach Aufstellung der obigen Definition im Herbst 1921 (vgl. K. MENDER, Monatshefte f. Math. u. Phys., 33, S. 160) fand ich, dass L. E. J. BROUWER in einem kurzen Aufsatz (Crelles Journal 142 (1913), vgl. auch Amsterdamer Akademieberichte 1923) für Kontinua eine Definition der Dimension formuliert hatte, die mit der obigen äquivalent ist. Für Teilmengen von kompakten metrischen Räumen hat auch P. URYSOHN unabhängig von mir die obige Dimensionsdefinition gegeben (vgl. Comptes Rendus, 4. September 1922).

Eine ausführliche Darstellung der im folgenden angedeuteten Dimensionstheorie ist bei den Monatsheften f. Math. u. Phys. Bd. 34 im Druck.

dann kann eine auf $m' = A(m)$ sich zusammenziehende Folge $\{U'_n(m')\}$ ($n = 1, 2, \dots$) von Umgebungen konstruiert werden, so dass, wenn B'_n die Begrenzung von $U'_n(m')$ bedeutet, für alle n gilt: $B'_n M' < A(B_n M)$. Berücksichtigt man dies, so folgt die Invarianz der Dimensionszahl durch vollständige Induktion. — Eineindeutige stetige Abbildungen (ohne stetige Umkehrung) können die Dimension unter Umständen erhöhen.

In die *Struktur* der n -dimensionalen Mengen können wir in mehrfacher Hinsicht Einblick gewinnen. Zunächst bildet die Menge jener Punkte einer Menge M , in denen M höchstens n -dimensional ist, für jedes n den Durchschnitt von M mit einem G_δ (so nennen wir den Durchschnitt abzählbar vieler offener Mengen). Der Satz folgt unmittelbar daraus, dass die Menge M_k Durchschnitt von M mit einer offenen Menge ist, wenn M_k definiert ist als Menge aller Punkte m von M mit folgender Eigenschaft: Es existiert eine Umgebung von m mit einem Durchmesser $< \frac{1}{k}$, deren Begrenzung mit M einen höchstens n -dimensionalen Durchschnitt hat. Die Menge aller Punkte von M , in denen M mindestens n -dimensional ist, bildet für jedes n den Durchschnitt von M mit einem F_σ (so nennen wir die Vereinigung abzählbar vieler abgeschlossener Mengen). Die Menge aller Punkte von M , in denen M genau n -dimensional ist, bildet daher für jedes n den Durchschnitt von M mit einem G_δ und einem F_σ , also eine einfach zu beherrschende Borelsche Menge in M .

Andere Struktursätze für abgeschlossene Mengen ergeben sich aus dem einfachen Überdeckungssatz: Ist M ein kompakter F_σ und ist jedem Punkt von M eine auf den betreffenden Punkt sich zusammenziehende Folge von Umgebungen zugeordnet, dann kann aus der Gesamtheit der so vorliegenden Umgebungen eine (eventuell endliche) Folge mit (falls sie unendlich ist) gegen Null konvergenten Durchmessern herausgegriffen werden, in deren Vereinigung M enthalten ist. Insbesondere können aus diesem Satz durch vollständige Induktion simultan die beiden folgenden Sätze abgeleitet werden: 1. Ist eine n -dimensionale Menge M abgeschlossen und sind ihre sämtlichen beschränkten Teile kompakt (wir nennen eine solche Menge der Kürze halber: total vollständig), dann ist die Menge aller Punkte, in denen M n -dimensional ist, nicht in einer abgeschlossenen weniger als n -dimensionalen Menge enthalten. 2. Ist die Vereinigung A abzählbar vieler abgeschlossener höchstens n -dimensionaler Mengen total vollständig, dann ist A höchstens n -dimensional.

Über die *Vereinigung abgeschlossener Mengen* gilt weitergehend folgender Satz: In einem total vollständigen Raum ist die Vereinigung

A von abzählbar vielen abgeschlossenen höchstens n -dimensionalen Mengen $\{A_k\}$ ($k=1, 2, \dots$) höchstens n -dimensional. Ist a ein Punkt von A , $V(a)$ eine vorgelegte Umgebung, so konstruieren wir eine Folge $\{U_k(a)\}$ ($k=1, 2, \dots$) von Umgebungen, deren Durchschnitt eine Umgebung $U(a) < V(a)$ ist. Dabei richten wir die Konstruktion so ein, dass, wenn B_k die Begrenzung von $U_k(a)$ bezeichnet, für jedes k die Menge $B_k \cdot \sum_{i=1}^k A_i$ höchstens $(n-1)$ -dimensional ist

und dass, für jedes $l > k$, B_l mit $\sum_{i=1}^k A_i$ keine anderen Punkte gemein

hat, als B_k . Wir sorgen sogar dafür, dass, für $l > k$, $B_l - B_k \cdot \sum_{i=1}^k A_i$ in

einer zu $\sum_{i=1}^k A_i$ fremden offenen Menge gelegen ist. Es ergibt sich

dann, dass die Begrenzung B von $U(a)$ mit A einen höchstens $(n-1)$ -dimensionalen Durchschnitt hat, womit bewiesen ist, dass A höchstens n -dimensional ist.

Es lässt sich schliesslich zeigen, dass, wenn M ein n -dimensionaler F_σ in einem Euklidischen Raum ist, zwischen je zwei offenen Mengen, deren eine nebst ihren Häufungspunkten in der anderen enthalten ist, eine dritte offene Menge sich einschalten lässt, mit deren Begrenzung M einen höchstens $(n-1)$ -dimensionalen Durchschnitt hat. — Aus dem Umstand, dass der angeführte einfache Überdeckungssatz für Mengen, die nicht F_σ sind, im allgemeinen versagt, ergeben sich grosse Schwierigkeiten für die Untersuchung der Struktur n -dimensionaler Mengen, die nicht abgeschlossen und auch nicht F_σ sind¹⁾. Diese Schwierigkeiten können behoben werden, wenn wir neben den n -dimensionalen Mengen noch „äusserlich n -dimensionale“ Mengen betrachten, die so definiert sind: Eine Menge M heisst äusserlich n -dimensional, wenn n die kleinste Zahl ist von folgender Eigenschaft: Zu jedem Punkt von M° (der abgeschlossenen Hülle von M , nicht bloss zu jedem Punkt von M !) existiert eine auf den betreffenden Punkt sich zusammenziehende Folge von Umgebungen, mit deren Begrenzungen M äusserlich höchstens $(n-1)$ -dimensionale Durchschnitte hat. Äusserlich (-1) -dimensional und äusserlich höchstens (-1) -dimensional ist die leere Menge und nur diese.

Eine Reihe weiterer Sätze, die bei unserem Ausgangspunkt von grundlegender Bedeutung sind, ergibt sich, wenn wir einen *Euklidischen*

¹⁾ Vgl. K. Menger, Einige Überdeckungssätze der Punktmengenlehre, im Druck bei den Wien. Ber., wo die nulldimensionalen Mengen ausführlich behandelt werden.

Raum zu Grunde legen. Durch vollständige Induktion bestätigt man unter Heranziehung von Intervallfolgen, dass der R_n höchstens n -dimensional ist, womit gleichzeitig erwiesen ist, dass unsere Definition jeder Menge eines Euklidischen Raumes eine Dimension zuordnet. Um zu zeigen, dass der R_n mindestens n -dimensional ist, haben wir nachzuweisen, dass die Begrenzung jeder beschränkten offenen Menge des R_n mindestens $(n-1)$ -dimensional ist. Nun hat LEBESGUE (Fund. Math. II, S. 274) gezeigt, dass bei jeder Zerlegung einer solchen Begrenzung in endlich viele hinlänglich kleine abgeschlossene Teilmengen ein Punkt existiert, der mindestens n von den Teilmengen gemein ist. Wir müssen also nachweisen, dass eine beschränkte abgeschlossene, höchstens $(n-2)$ -dimensionale Menge eines Euklidischen Raumes für jedes $\varepsilon > 0$ in endlich viele abgeschlossene Teilmengen zerlegt werden kann, deren Durchmesser sämtlich kleiner als ε sind, und so, dass je n von diesen Teilmengen keinen Punkt gemein haben. Diese Behauptung ist in dem weitergehenden Hilfssatz enthalten: Ist A eine beschränkte abgeschlossene höchstens $(n-1)$ -dimensionale Menge eines Euklidischen Raumes und $\varepsilon > 0$ eine vorgegebene Zahl, dann existieren endlich viele zu je zweien fremde offene Mengen U_1, U_2, \dots, U_i , so dass: 1. jeder Punkt von A innerer Punkt der Vereinigung der abgeschlossenen Hüllen der U_k ist. 2. dass die Durchmesser aller U_k kleiner als ε sind. 3. dass je $n+1$ von den abgeschlossenen Hüllen der U_k keinen Punkt gemein haben. Der Beweis dieses Hilfssatzes ergibt sich aber durch vollständige Induktion. Damit ist gezeigt, dass der R_n und dass die Mengen, die im R_n einen offenen Teil enthalten, sowie die topologischen Bilder dieser Mengen n -dimensional im Sinne unserer Definition sind — Sätze, welche die Berechtigung unserer Definition beweisen. Äquivalent mit diesen Sätzen ist ferner die Behauptung, dass das Komplement einer beschränkten abgeschlossenen n -dimensionalen Menge zum R_{n+2} zusammenhängend ist. Für $n=0$ geht dieser Satz in den von SIERPINSKI über, dass das Komplement einer abgeschlossenen zusammenhangslosen Menge zum R_n ($n \geq 2$) zusammenhängend ist. Für $n=1$ und $n=2$ besagt er, dass das Komplement der Kurven zum R_n ($n \geq 3$) und das der Flächen zum R_n ($n \geq 4$) zusammenhängend ist.

Der Satz von der n -Dimensionalität der offenen Mengen des R_n lässt sich aber auch umkehren: Jede n -dimensionale Menge des R_n enthält einen offenen Teil. Bezeichnen wir der Kürze halber eine den Punkt m enthaltende topologische Kugel, deren Begrenzung mit jedem von m ausgehenden Halbstrahl genau einen Punkt gemeinsam hat, als reguläre Umgebung von m , dann gilt sogar folgender Satz: Ist

M eine Menge des R_n und existiert zugleich ein Punkt m und eine Zahl $\varrho > 0$, so dass der Durchschnitt von M mit der Begrenzung jeder regulären Umgebung von $m < U(m; \varrho)$ $(n-1)$ -dimensional ist, dann enthält M einen offenen Teil. Der Beweis erfolgt durch vollständige Induktion nach n . Unter der Annahme, dass M keinen offenen Teil enthält, können wir schrittweise eine abzählbare zu M fremde Menge von Punkten konstruieren, welche in der Begrenzung einer regulären Umgebung von $m < U(m; \varrho)$ enthalten ist und in ihr dicht liegt. Diese Begrenzung hat mit M keinen in der Begrenzung offenen Teil gemein; also ist ihr Durchschnitt mit M nach dem für den R_{n-1} als bewiesen angenommenen Satz höchstens $(n-2)$ -dimensional, im Widerspruch mit der Voraussetzung. — Aus den bewiesenen Sätzen ergibt sich, dass die Begrenzung jeder beschränkten offenen Menge des R_n $(n-1)$ -dimensional ist. Schliesslich erlaubt der vorhin angedeutete Beweis die Definition der n -dimensionalen Mengen des R_n folgendermassen auszusprechen: Es sind die Mengen, für die ein Punkt m existiert, so dass der Durchschnitt von M mit der Begrenzung jeder hinlänglich kleinen regulären Umgebung von m $(n-1)$ -dimensional ist.

Mathematics. — “*Bemerkungen zum Beweise der gleichmässigen Stetigkeit voller Funktionen*”. By Prof. L. E. J. BROUWER.

(Communicated at the meeting of September 27, 1924).

§ 1.

Die Entstehungsweise der beim im Titel angeführten Beweise (diese Proceedings 27, S. 189—193) im zweiten Absatz von S. 190 eingeführten wohlgeordneten Spezies $f_{sn_1 \dots n_r}$ erfordert folgende nähere Erörterung:

Die Beweisführung $k_{sn_1 \dots n_r}$ bildet eine wohlgeordnete Spezies, von der jedes Element entweder ein F -Schluss oder ein ζ -Schluss oder ein Nullelement ist. Mittels transfiniter Induktion ergibt sich aber die Existenz eines Gesetzes, das einem beliebigen, die Versicherbarkeit eines Elementes $F_{mm_1 \dots m_g}$ von σ feststellenden, Elemente e von $k_{sn_1 \dots n_r}$ ein solches in $k_{sn_1 \dots n_r}$ nicht höher als e geordnetes Element e_0 von $k_{sn_1 \dots n_r}$ zuordnet, das erstens ein F -Schluss ist und zweitens ebenfalls die Versicherbarkeit von $F_{mm_1 \dots m_g}$ feststellt ¹⁾. Wenn also $k'_{sn_1 \dots n_r}$ die Beweisführung vorstellt, die aus $k_{sn_1 \dots n_r}$ entsteht, wenn alle Elemente von $k_{sn_1 \dots n_r}$, welche ζ -Schlüsse sind oder die Versicherbarkeit eines Elementes von τ feststellen, durch Nullelemente ersetzt werden, so ist (wie sich wiederum mittels transfiniter Induktion herausstellt) auch $k'_{sn_1 \dots n_r}$ eine zur Herleitung der Versicherbarkeit von $F_{sn_1 \dots n_r}$ ausreichende Beweisführung.

¹⁾ Diese Eigenschaft kann auch wie folgt mittels des im § 3 zitierten (übrigens ebenfalls auf transfiniter Induktion beruhenden) *Haupttheorems der Theorie der wohlgeordneten Spezies* hergeleitet werden: Es stelle e die Versicherbarkeit von $F_{mm_1 \dots m_g}$ als ζ -Schluss fest und es sei S die Vereinigung der Elemente $F_m, F_{mm_1}, \dots, F_{mm_1 \dots m_g}$ von σ . Dann geht aus $k_{sn_1 \dots n_r}$ auf Grund des Haupttheorems ein Gesetz hervor, das eine nach der Wohlordnung in $k_{sn_1 \dots n_r}$ absteigende Folge $e, e', e'', \dots, e^{(t)}$ von Elementen von $k_{sn_1 \dots n_r}$ bestimmt, zu der eine solche Folge $\eta, \eta', \eta'', \dots, \eta^{(t)}$ von Elementen von S gehört, dass $e^{(\nu)}$ für $0 \leq \nu < t$ die Versicherbarkeit von $\eta^{(\nu)}$ aus (wenigstens unter andern aus) derjenigen von $\eta^{(\nu+1)}$, $e^{(t)}$ aber die Versicherbarkeit von $\eta^{(t)}$ aus derjenigen von einem oder mehreren nicht zu S gehörenden Elementen von μ herleitet. Dann aber ist $e^{(t)}$ ein die Versicherbarkeit von $\eta = F_{mm_1 \dots m_g}$ feststellender F -Schluss.

Wenn wir nun in $k'_{sn_1 \dots n_r}$ jedes Element, das ein r -Schluss ist, durch das Element von σ , dessen Versicherbarkeit es feststellt, ersetzen, so entsteht die a. a. O. eingeführte wohlgeordnete Spezies $f_{sn_1 \dots n_r}$, welche aber wegen des Auftretens eventueller Nullelemente *nicht* notwendig ein *erstes* nicht-verschwindendes Element zu besitzen braucht. Ein solches *erstes* Element wird also a. a. O. Z. 24 mit Unrecht erwähnt, die Beweiskraft des betreffenden Passus wird jedoch hierdurch nicht beeinträchtigt, weil eben die jetzt gegebene nähere Beschreibung der Entstehungsweise von $f_{sn_1 \dots n_r}$ für die Herleitung der Versicherbarkeit der Elemente dieser wohlgeordneten Spezies mittels $k'_{sn_1 \dots n_r}$ ohne weiteres die Ausschaltung des ξ -Schlusstypus impliziert.

An die obige Erörterung wird a. a. O. Anschluss erhalten, wenn daselbst, S. 190 Z. 31 und Z. 35, $k_{sn_1 \dots n_r}$ durch $k'_{sn_1 \dots n_r}$ ersetzt wird.

Hinsichtlich des a. a. O. im dritten Absatz von S. 190 angegebenen Verfahrens ist weiter zu bemerken, dass (wie sich mittels transfiniter Induktion in bezug auf $f_{sn_1 \dots n_r}$ ergibt) ein in $f_{sn_1 \dots n_r}$ öfters auftretendes Element $F_{mm_1 \dots m_g}$ von σ dabei auf Grund des angeführten Verfahrens jedesmal in dieselbe wohlgeordnete Spezies übergeht. Hieraus folgt insbesondere, dass die durch Theorem 1 (a. a. O. S. 191) erklärte wohlgeordnete Spezies S eindeutig bestimmt ist.

§ 2.

Das *Prinzip der transfiniten Induktion*, von dem in diesem Zusammenhang fortwährend die Rede ist, kann wie folgt formuliert werden:

Wenn für eine wohlgeordnete Spezies W und eine Eigenschaft E feststeht:

1. dass E erfüllt ist für das erste (verschwindende oder nicht verschwindende) Element von W ,

2. dass E , wenn sie erfüllt ist für die konstruktive Unterspezies w von W , ebenfalls erfüllt ist für $w + e$, wo e eine Urspezies von W vorstellt,

3. dass, wenn bei einer beliebigen in einer Summierung $u + w_1 + w_2 + w_3 + \dots = w$ bestehenden zweiten erzeugenden Operation von W für jedes ganze positive v aus der Erfüllung von E für $u + w_1 + \dots + w_{v-1}$ die Erfüllung von E für $u + w_1 + \dots + w_v$ folgt, aus der Erfüllung von E für u die Erfüllung von E für w folgt, so ist E erfüllt für W .

Für die Herleitung von Theorem 1 (a. a. O. S. 191) mittels

dieses Prinzips ist genau genommen die a. a. O. geschehene Einführung von $f_{s_{n_1} \dots n_r}$ nicht nötig; auch transfinite Induktion in bezug auf $k'_{s_{n_1} \dots n_r}$ oder sogar in bezug auf $k_{s_{n_1} \dots n_r}$ führt zum Ziel.

§ 3.

Das sich a. a. O. als besonderer Fall von Theorem 1 ergebende Theorem 2 kann auch unabhängig von Theorem 1 bewiesen werden, und zwar mittels Anwendung des folgenden *Haupttheorems der Theorie der wohlgeordneten Spezies*:

Ein Gesetz, welches in einer wohlgeordneten Spezies W ein Element e' bestimmt, und jedem schon bestimmten Elemente $e^{(v)}$ entweder die Hemmung des Prozesses, oder ein in W niedriger als $e^{(v)}$ geordnetes Element $e^{(v+1)}$ zuordnet, bestimmt sicher eine natürliche Zahl n und ein zugehöriges Element $e^{(n)}$, dem es die Hemmung des Prozesses zuordnet.

Nehmen wir nämlich an, dass F_s zu σ gehört. Einem die Versicherbarkeit von F_s feststellenden Elemente e von k_s ist ein in k_s nicht höher als e geordnetes Element ${}_oe$ von k_s zugeordnet, das als F -Schluss die Versicherbarkeit von F_s feststellt; dem Elemente ${}_oe$ von k_s für jedes zu σ gehörige $F_{s\alpha}$ ein in k_s niedriger als ${}_oe$ geordnetes Element e'_α von k_s , das die Versicherbarkeit von $F_{s\alpha}$ feststellt; jedem e'_α ein in k_s nicht höher als e'_α geordnetes Element ${}_oe'_\alpha$ von k_s , das als F -Schluss die Versicherbarkeit von $F_{s\alpha}$ feststellt; jedem ${}_oe'_\alpha$ für jedes zu σ gehörige $F_{s\alpha\beta}$ ein in k_s niedriger als ${}_oe'_\alpha$ geordnetes Element $e''_{\alpha\beta}$ von k_s , das die Versicherbarkeit von $F_{s\alpha\beta}$ feststellt; usw. Im Falle einer finiten Menge M gibt es unter den e'_α ein in k_s am höchsten geordnetes Element e' ; unter den $e''_{\alpha\beta}$ ein in k_s am höchsten geordnetes Element e'' ; usw. Weiter gibt es ein Gesetz, das das Element e' dem Elemente e , das Element e'' dem Elemente e' , das Element e''' dem Elemente e'' , usw. zuordnet, wobei jedesmal $e^{(v+1)}$ in k_s niedriger als $e^{(v)}$ geordnet ist; nach dem oben zitierten Haupttheorem der Theorie der wohlgeordneten Spezies bestimmt mithin dieses Gesetz eine natürliche Zahl n mit der Eigenschaft, dass $v < n$ für jedes $e^{(v)}$. Dies aber ist nur dann möglich, wenn jedes Element von μ , das eine Verlängerung von F_s bildet und mindestens $z = n + 1$ Wahlen enthält, zu τ gehört. Von der letzteren Eigenschaft ist aber das zitierte Theorem 2 eine unmittelbare Folge ²⁾.

²⁾ Der obige Beweis lässt sich formal noch etwas kürzen, indem man, statt mit k_s , mit f_s operiert.

Physiology. — "*On the Heredity of Keratoconus and other Deformities of the Eye.*" By Prof. J. VAN DER HOEVE.

(Communicated at the meeting of June 28, 1924).

Since the revival of MENDEL's law, the subject of heredity has attracted a large amount of attention among biologists.

No property of plants, animals or men, but was subjected to the question: is it hereditary or not? and if it is: does it follow MENDEL's law?

In botany this may be readily ascertained, as there is unlimited scope for experimentation under the most favourable conditions for the development of the property under consideration.

Animals also may be experimented on for this matter, but whoever tried to do so is fully aware, that great difficulties block the way, especially when higher animals are concerned.

With man experimentation in this domain is precluded altogether; one has to confine oneself to the material that nature bestows upon us in the families, in which some property or deviation is hereditary. An attempt is then made to establish a pedigree of these families, as far as this is possible under the circumstances.

Ophthalmologists have been making such pedigrees for many years, and have not confined themselves to physiological properties, but they have also concerned themselves about morbid anomalies such as glaucoma, cataract, keratoconus, etc.

For most of these diseases it is easy to establish their heredity; it is much more difficult, however, to ascertain whether the anomaly follows numerical rules such as MENDEL's. All sorts of difficulties will crop up:

1°. The material differs largely as to quantity: it is very abundant in cases of myopia, mostly very scanty in recessive diseases, so that we often have to content ourselves with a few pedigrees.

2°. Outside influences may come into play so as to retard, nay even to inhibit the appearance of that hereditary anomaly in spite of the presence of the hereditary factor.

3°. The hereditary anomaly does not always appear already at birth, it may come forth even very late in life, the so-called homochronous inheritance; so senile cataract occurs late; generally it does not get

ripe before about the 65th year; it is evident that many individuals of a family in which this anomaly occurs, will have died before it has been established, so that the numerical relations are considerably modified.

4°. The hereditary deviation does not always result directly from the hereditary factor, but may be its remote consequence, so that influences may play a role here that otherwise would not assert themselves.

5°. It may very well be possible that an hereditary property does not arise from one and the same hereditary factor alone, but from a combination of several of them, so that, while the hereditary factor may obey certain laws, the deviation cannot be expected to do so on that score.

6°. One and the same hereditary anomaly may occasionally be brought on by different hereditary factors and by outside influences. Some instances taken from ophthalmology may illustrate this.

Boeophthalmos i.e. the presence of an abnormally large eye, may occur:

a. as a direct outcome from the hereditary factor; prodigious growth of the eye, so that a large, but for the rest quite normal, eye is developed.

b. as an indirect consequence of the hereditary factor: the absence of the canalis Schlemmii. This may induce a rise of pressure with secondary dilatation of the eye-ball. Notwithstanding the presence of this hereditary factor the eye may retain its normal size, when the other efferent ducts duly supply the place of Schlemm's canal.

c. as a mediate result from other hereditary factors, such as bad condition of the other efferent ducts for the liquid; too copious a secretion of the eye-liquid and the like.

d. caused by outside influences (so-called peristatical), e.g. iritis with seclusio pupillae and consequent rise of pressure.

It is evident, therefore, that boeophthalmos is engendered by various hereditary factors and peristatically, while it may be absent even when an hereditary factor is present. Moreover, already in intra uterine life boeophthalmos may bring on atrophy or perforation of the eye, so that the individual is not born with an large eye.

It may be readily understood that under these circumstances the numerical rules can hardly be recognized.

Tetany-cataract, i.e. a cataract as a complication in tetany, may be hereditary. As to its genesis this still unproved theory has been advanced: The primary cause is anomaly of the glandulae parathyreoideae which evokes spasm, a.o. of the ciliary

muscle; these spasms injure the epithelium of the ciliary body by which the secreted liquid, which serves for nutrition of the lens, is modified, so that cataract is originated. It is obvious that the presence of the hereditary factor: anomaly of the glandulae parathyreoideae, does not at all induce tetany-cataract in all cases. The same degree of this factor will produce spasm of the ciliary muscle in one individual, not in the other; the same spasm of this muscle will cause lesion of the epithelium with one individual, but not with the other; the same modification in the eye-fluids will elicit cataract in one lens, not in the other. So if the hereditary factor acts according to fixed rules, these rules will not of necessity be observable in the cataract appearing as its remote effect. It is more a matter of surprise that the inheritance of cataract can be established than that it does not follow the rules.

Nowadays senile cataract is also considered as being hereditary, while there is a tendency to ignore other influences on its origin. But I know a family of whom the members that live in Europe, have to be operated upon for senile cataract at the age of 65, whereas the members that have long lived in India came to operation at the age of 47. Heredity alone does not account for this.

In my theory on the etiology of cataract one of the chief causes is the modification of the ciliary epithelium under the influence of light with many ultraviolet rays. Strong lightsources of that kind will reveal in animals a modification of the eye-liquids inside of a quarter of an hour; a more prolonged radiation will elicit severe inflammation of the corpus ciliare.

In my judgment the ciliary epithelium so well guarded by outside walls is injured from within, the lens being optically heterogeneous, not optically empty, so that part of the light is dispersed diffusely, also in the direction of the corpus ciliare. This dispersed light contains comparatively many ultraviolet rays, because the dispersion is always stronger for rays of a short wavelength. According to RAILEIGH'S law the dispersion is even inversely proportional to the fourth power of the wavelength, so for ultraviolet of 300 μ about 50 times stronger than for red of 800 μ .

Since a large dosis of it evokes acute inflammation, it may be that radiation by sun-, and day-light prolonged for years, brings about an inappreciable change, which by modifying the nutritive liquid for the lens may excite senile cataract.

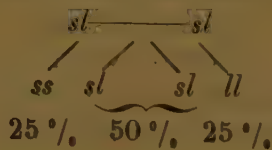
Now the hereditary factor may be marked opalescence of the lens, which differs very much individually. Besides this optical heterogeneity light with many ultraviolet rays is also required, so that

we see that, in the family alluded to, the members in India, who are exposed to much light, contract a cataract much earlier than the members living in Europe. A still more effective protection of the eye would perhaps retard the appearance of the disease so long that it would not reach maturity.

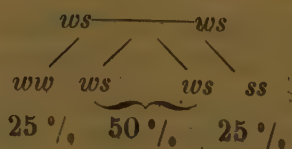
A disease elicited by more than one hereditary factor is myopia. It is brought about by a disproportion between the refraction, which is relatively too strong, and the axis of the eye, which is relatively too long.

Since the investigation carried out by CORN at Breslau in the middle of last century, myopia has chiefly been ascribed to working at short distance-vision. In that time fears were even entertained that through this sort of work the intellectuals would eventually incur blindness. About ten years ago STEIGER has called attention to the fact that the long since recognized hereditary moment in the appearance of myopia was largely underrated. He believes that the length of the axis and corneal refraction are hereditary factors, each in their own way. An examination of 50000 corneae brought to light that the corneal refraction in emmetropic persons is liable to considerable physiological variations and STEIGER reasoned in this way: if an emmetropic eye has a cornea that refracts 5 D. more than normal, the length of the axis must be proportionately shorter; if the refraction is 5 D. weaker than normal, the axis must be longer in the same proportion.

Now let us take the case of two emmetropic persons with a normal length of the eye-axis, because a long and a short hereditary factor are present in them, with whom also the refraction of the cornea is normal, because a strong and a weak hereditary factor are present, which counterbalance each other. If these persons are united in marriage, the eye-axis of the offspring will, according to MENDEL'S law be short in 25 %, long in 25 %, and normal in 50 %:



For the corneal refraction the values are 25 %, strong, 25 %, weak and 50 %, normal:



In $\frac{1}{16}$ of the offspring the weak refraction will coincide with the short eye-axis, so that an intense hypermetropia will manifest itself; in $\frac{1}{16}$ the strong refraction will coincide with the long eye-axis, consequently a high degree of myopia will appear.

This reasoning induces us to assume that coincident with physiological varieties in the corneal refraction of 5 D. on either side of the normal physiological varieties in the refraction may be expected from 10 D. hypermetropia to 10 D. myopia.

In this case the laws of heredity will not be very conspicuous, as far as myopia is concerned, though their course may be followed for corneal refraction.

The great significance ascribed to heredity has given rise to a threatening danger, viz. that the presence of hereditary deviations is looked upon as a mere fatalistic fact, and no effort is made to prevent or relieve the disease. This I consider to be a mistake. Since outward circumstances can play such an active part in the occurrence of the deviations, we may exert some influence by modifying those outward circumstances. In cases e.g. of incipient myopia it is incumbent on us to improve as much as possible the circumstances that may also be answerable for the lengthening of the axis of the eye.

From the foregoing it will be inferred, that we have little reason to expect an hereditary deviation in man to follow MENDEL's law. It should, moreover, be borne in mind that in plants and animals this law holds only for averages out of large numbers.

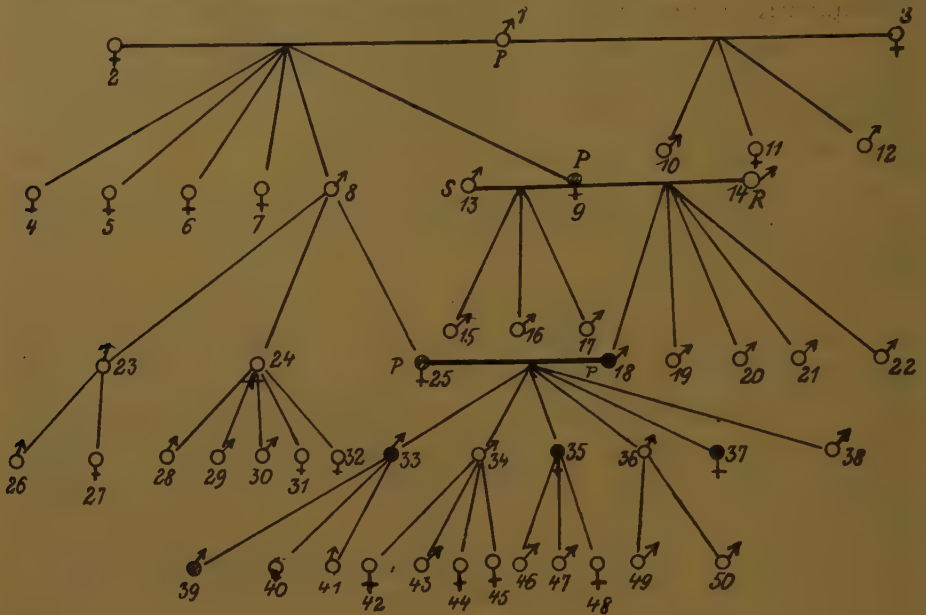
This may be expected in man with some reason, in those hereditary anomalies which are congenital, not subject to the influence of external circumstances, and cannot be caused peristatically, e.g. the typical redgreen-colourblindness. We can, indeed, clearly distinguish in this affection the type of the recessive hereditary anomaly limited by sex.

A rare instance of deformity is keratoconus, in which at a certain age, most often after the 10th year, sometimes not before the 40th year, the spherical shape of the cornea is transformed into a conical one, consequent on a peculiar decrease of the resistance of this tissue.

Whether this deformity is hereditary or not, is still a subject of uncertainty. The annexed pedigree (see fig. page 652) illustrates the keratoconus as probably a typically recessive disease, whose frequency will increase through marriages between blood-relations.

N°. 33 of the pedigree I have known since twenty years suffering from bilateral keratoconus. Inquiries and investigations made

out that not only two sisters were affected with the same deformity, but also the father N°. 18, who had been operated upon for it by Prof. DOYER at Leyde already in 1869. DOYER seemed already



then to have surmised the possibility of heredity. Anyhow, in the *historia morbi* he noted down the visual acuity of the man's parents N°. 9 and N°. 14, for the mother respectively $\frac{2}{3}$, and $\frac{2}{3}$, of the normal, for the father $\frac{2}{3}$, and normal. It would seem then as if the man had inherited the deformity from the mother; although in her case DOYER did not diagnose keratoconus, yet he notes that the fundus oculi is abnormally elongated. It is certain, therefore, that this woman was astigmatic, was perhaps affected with keratoconus. Now the question was: why half of the children, procreated by 18, suffered from keratoconus? On inquiry it appeared that this man had married a cousin german (25). This family-marriage accounted for the increase of the cases of keratoconus; but husband and wife bore the same familyname; it seemed, therefore, that it was a paternal relationship on both sides, whereas we supposed that the husband had acquired the deformity from the mother's side.

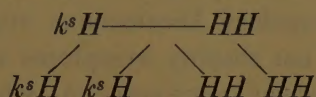
Inquiry and painstaking research-work at the registrar's office yielded the following results:

Mrs. P., N°. 9, was married to Mr. S. N°. 13, by whom she had three children; she wanted to be divorced. S. refused and she then lived in free union with R N°. 14, by whom she had five children (18—22). Legally she was obliged to give notice of the births of

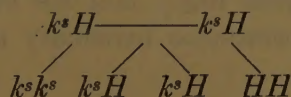
these five children as the legitimate husband's, but she had no mind to do this. She could not have them registered as the real father's (R), so she gave them her own name. This illegal action of N°. 9 caused that 25 and 18 had the same name, although their kinship originated from the father of the one and the mother of the other.

This is an illustration of another difficulty in establishing numerical rules for heredity in man. It is not only that a single instance of conjugal infidelity may render our pedigree valueless, but even a permanent illegitimate relation, such as that between 9 and 14, may escape our notice, if the woman had had her children christened S, according to the law.

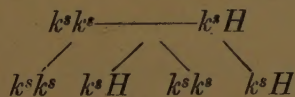
When surveying the whole pedigree it occurs to me, that the case under consideration is a typically recessive anomaly. In one of the progenitors 1 and 2 the hereditary factor keratoconus was latent. He marries a healthy woman, and according to MENDEL's law transmits the latent disease to 50 % of the children, in this case to



8 and 9. 8 transmits it again as a latent disease to half of his children, decidedly to 25. 9 has 5 children by 14. Now we have to assume that also R harbours a latent keratoconus factor. Then according to MENDEL's law 25 % of R's issue will have homozygous



manifest keratoconus. This tallies with the fact that one of the five children, N°. 18, has keratoconus. N°. 18 marries N°. 25. Of their



issue, according to MENDEL's law, 50 % must reveal homozygous i.e. manifest and 50 % heterozygous i.e. latent keratoconus. This agrees with the fact that 33, 35 and 37 have keratoconus.

Of the next generation none of the children will suffer from homozygous keratoconus; all the children of the keratoconus-sufferers 33, 35 and 37, will have heterozygous latent keratoconus; of the children of the sufferers from heterozygous keratoconus, 34, 36, 38 only the one half, while the other half will be quite normal in this respect.

If in adult life one of the children of this generation evinces manifest keratoconus, this must be ascribed to another keratoconus factor in the other parent of this child, which factor may be searched for in this parent's relations. This research will by no means always be successful, since a recessive hereditary factor may remain latent in a family for 100 years or more, unless a member of this family marries somebody who harbours a similar recessive hereditary factor.

Two more questions remain to be answered:

1°. How is it that 9 and 25 are astigmatic? It is possible that the combination of the hereditary factor keratoconus with a healthy factor, does not produce keratoconus, but that the healthy factor does not absolutely dominate, so that astigmatism appears. If this be true most of the children of the generation 39—50 must become astigmatic.

2°. What is the hereditary factor here? Is it a direct deficiency of resistance of the cornea, or is it, as SIEGRIST believes, a morbid „Anlage” of the glandulae thyreoideae which engenders dystheroidismus, from which a secondary keratoconus might arise. The patients examined for this did not display symptoms of dystheroidismus, but in 33 and 39 the ABDERHALDEN-reaction, which was tested for all the endocrine glands, was highly positive for the thyroid only.

In cases where dystheroidismus is presumably the hereditary factor, the process may be arrested by preparations from endocrine glands.

To sum up, it is most likely that we have to do here with a typically recessive homochronous hereditary anomaly.

